

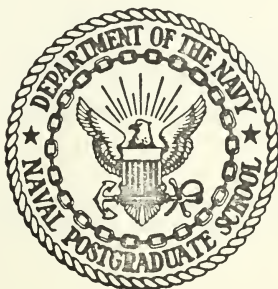
SINGULAR PERTURBATION PROBLEMS IN
EARTH-MOON SPACE

by

Vernon Curtis Gordon

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THESIS

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IN EARTH-MOON SPACE

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Vernon Curtis Gordon

June 1970

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Singular Perturbation Problems
in Earth-Moon Space

by

Vernon Curtis Gordon
Lieutenant (junior grade), United States Navy
B.A.E., Auburn University, 1968

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ABSTRACT

A program was written for the IBM 360 system of the Naval Postgraduate School Computer Facility to apply the methods of singular perturbation theory to earth-moon trajectories.

Verification of results against previous work was carried out and it was found that agreement could be attained only for energy parameter values of 0.707 or less. No solution for higher values could be found.

The analysis of three-dimensional orbits was then conducted within this restricted range to show the merit of singular perturbation theory as an initial design tool.

TABLE OF CONTENTS

I.	INTRODUCTION -----	9
II.	THEORY -----	11
	A. EXAMPLE INNER AND OUTER EXPANSION MATCHING PROCEDURE -----	11
	B. NATURE OF THE PROBLEM -----	17
	C. FORMULATION OF THE RESTRICTED THREE-BODY PROBLEM -----	22
	D. OUTER EXPANSION -----	27
	E. INNER EXPANSION -----	30
	F. MATCHING OF INNER AND OUTER EXPANSIONS -----	33
III.	COMPUTER PROGRAM -----	38
IV.	DISCUSSION OF RESULTS -----	40
	A. VERIFICATION OF COMPUTER RESULTS -----	40
	B. PARAMETRIC STUDY IN THREE DIMENSIONAL EARTH-MOON SYSTEM -----	40
V.	CONCLUSIONS AND RECOMMENDATIONS -----	44
	APPENDIX A: FIGURES AND GRAPHS -----	46
	COMPUTER OUTPUT SAMPLE CASE -----	63
	COMPUTER PROGRAM LISTING -----	67
	REFERENCES -----	73
	INITIAL DISTRIBUTION LIST -----	74
	FORM DD 1473 -----	75

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LIST OF FIGURES

Figure

1	Ames, Van Dyke Boundary Layer Problem -----	46
2	Flow Chart of Matching Principle -----	47
3	Example Earth-Moon Hyperbolic Encounter Trajectory -----	48
4	Three-Body Geometry -----	49
5	Earth-Moon Coordinate Systems for Restricted 3-Body Problem -----	50
6	Angular Momentum Correction -----	51
7	Angular Momentum Relative to the Moon -----	51
8	Distances of Closest Encounter with the Moon ----	52
9	Perigee Height vs Lambda for Values of Rho -----	53
10	Perigee Velocity vs Lambda for Values of Rho ----	53
11	Inclination vs Closest Approach for Varying Energy, All Other Parameters Constant -----	54
12	Inclination vs Closest Approach for Varying Angular Momentum, All Other Parameters Constant -----	55
13	Inclination vs Closest Approach for Varying Phase Constant (T_1), All Other Parameters Constant -----	56
14	Earth-Moon Orbit 3-D Geometry -----	57
15	Moon Passage 3-D Geometry -----	57
16	ω vs Closest Approach for Varying Angular Momentum at Small Inclination, All Others Constant -----	58
17	ω vs Closest Approach for Varying Energy at Small Inclination, All Others Constant -----	59
18	ω vs Closest Approach for Varying Phase Constant at Small Inclination, All Others Constant -----	60

19	Program Flow Chart Part A -----	61
20	Program Flow Chart Part B -----	62

TABLE OF SYMBOLS

Arabic Symbols

a	Semi major axis of earth elliptic orbit
\bar{a}	Semi major axis of moon hyperbolic orbit
b	Semi minor axis of ellipse
D	Earth-moon separation distance
DOCA	Distance of closest approach to moon
e	Eccentricity of outer ellipse
E	Eccentric anomaly
G	Universal Gravitational Constant
h_e	Energy of elliptic orbit about earth
\bar{h}	Energy of hyperbolic orbit near moon
i	Inclination from earth-moon plane at earth
\bar{i}	Inclination from earth-moon plane at moon
l_e	Angular momentum of elliptic orbit near earth
\bar{l}	Angular momentum of hyperbolic orbit near moon
m	Mass
p_e, q_e	Position parameters at earth
p, q	Position parameters at moon
\bar{r}	Position vector
T	Phase constant
U	Velocity in x direction
V	Velocity in y direction
x	Independent outer variable

X and \bar{x} Independent inner variables
 y and t Dependent outer variables
 \bar{y} and \bar{t} Dependent inner variables

Greek Symbols

γ Integral of perturbation equation in t
 δ Integral of perturbation equation in y
 ξ Moon's x coordinate
 η Moon's y coordinate
 λ Non-dimensional angular momentum
 ρ Non-dimensional energy
 ψ Angle relative earth after encounter of moon
 Ω Line of nodes in xyz system
 $\bar{\Omega}$ Line of nodes in $\bar{x}\bar{y}\bar{z}$ system
 $\bar{\theta}$ Apse angle, moon
 ω Apse angle, earth

I. INTRODUCTION

With the advent of space travel and of complex satellite systems, a revival of celestial mechanics has appeared in modern science and engineering. In its present form, commonly referred to as orbital mechanics or astrodynamics, it combines the theories of astronomers and scientists such as Kepler and Newton with modern ideas and technology so that man may navigate his way through space.

Orbital mechanics problems grow in complexity as the number of bodies under consideration grow. In fact, they become so complex that only the problem of two-bodies may be truly solved. Solutions to forms of the three-body problem are accomplished under certain special conditions, but usually numerical approximations exist for these higher order problems. Szebehely [Ref. 1] states,

"Not only is the two-body problem solved -- and the meaning of 'solution' may be different for astronomers, engineers, and mathematicians -- but a general understanding exists regarding this dynamical system. The problem of three-bodies on the other hand is neither solved nor is the behavior of the dynamical system completely understood.

In order to cope with the problem of space navigation, many methods to determine the motion of a vehicle in space have been attempted. The use of two-body solutions has been used in the method of patched conic sections. Many perturbation techniques have been applied for two, three, four, and more bodies. The most common method of calculating

these trajectories is probably the use of the digital computer. The computer is used to solve the initial-or final-value differential equations of motion in numerical form and is often time consuming and expensive.

Due to the difficulty in finding solutions to trajectory problems, Lagerstrom and Kevorkian in References 2 and 3 formulated a method in which they attempt to treat the restricted three-body problem with a technique similar to that used in certain fields of fluid mechanics. This is the method of solution of singular perturbations by inner and outer expansions and represents an attempt to obtain a uniformly valid analytic expression for trajectory motion in the restricted three-body problem. The work of Lagerstrom and Kevorkian is aimed at giving a tool to more easily predict the effect of initial conditions on the motion of a vehicle as it becomes influenced by a third body in its field of space. This would allow the determination of "ballpark" data for use in more detailed solution programs at a considerable savings of time and expense.

In this paper, an attempt to use the singular perturbation method of Lagerstrom and Kevorkian to study effects of initial conditions on lunar trajectories is made. A brief discussion of Lagerstrom and Kevorkian's theory is followed by some numerical applications of this theory and the results are discussed.

II. THEORY

A. EXAMPLE INNER AND OUTER EXPANSION MATCHING PROBLEM

Before formulating the singular perturbation solution to the restricted three-body problem it is useful to look at a simple example of matched inner and outer asymptotic expansions. The example is basically that of Prandtl and has been used by Van Dyke [4], Ames [5] and others as a classical singular perturbation solution.

If an equation of the form

$$\mu X^{(n)}(t) + F[X, X', \dots, X^{(n-1)}, \mu] = 0$$

is written and a regular expansion is attempted, difficulty is expected since the equation obtained when $\mu = 0$ is of lower order than the original.

Consider the following equation

$$\mu f'' + f' = b \tag{1}$$

with initial conditions,

$$f(0) = 0 \qquad \text{and} \qquad f(1) = 1$$

The exact solution to this equation is

$$f(x; \mu) = (1-b) \left[1 - e^{-x/\mu} \right] / \left[1 - e^{-1/\mu} \right] + b x \tag{2}$$

If a regular perturbation expansion is attempted one may write

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \dots$$

$$f' = f'_0 + \mu f'_1 + \mu^2 f'_2 + \dots$$

$$f'' = f''_0 + \mu f''_1 + \mu^2 f''_2 + \dots$$

substituting into 1

$$\mu[f''_0 + \mu f''_1 + \mu^2 f''_2 + \dots] + f'_0 + \mu f'_1 + \mu^2 f'_2 + \dots = b$$

matching powers of μ implies

$$f'_0 = b$$

so that by integration

$$f_0 = bx + C \quad (3)$$

The boundary conditions cannot be satisfied unless $b = 1$.

If the boundary condition at $x = 0$ is dropped

$$f_0(x, \mu) = (1-b) + bx$$

since

$$f(1) = 1 = b(1) + c \rightarrow c = 1-b$$

therefore

$$f_0(x, \mu) = bx + (1-b) \quad (4)$$

This approximates the solution quite well except in the region adjacent $x = 0$ (commonly referred to as the boundary layer), see Figure 1.

Ames states that if terms are lost or boundary conditions discarded in the outer solution they must be included in an inner expansion which must be developed (i.e., this implies matching in the overlap region).

The asymptotic matching principle states that the m-term inner expansion of (the n-term outer expansion) = the n-term outer expansion of (the m-term inner expansion). See Figure 2 for a diagram of the calculation sequence applying this principle.

Returning to the present example, the first term is

$$f_0 = (1-b) + bx$$

for the outer expansion. The first term of the inner expansion is found by a "coordinate stretching." The dependent and the independent variables may be stretched together or separately using physical insight or trial and error, however, the stretching must include essential elements omitted in the outer expansion procedures and must be matchable to the outer expansion.

In the present example one may desire to stretch only the independent variable such that,

$$f(x, \mu) = F(X, \mu)$$

where

$$X = x/s(\mu)$$

Using this in the initial equation

$$\frac{d^2 F}{dX^2} + \frac{s(\mu)}{\mu} \frac{dF}{dX} = \frac{s^2(\mu)b}{\mu} \quad (5)$$

with

$$F(0) = 0, \quad F(1/s) = 1$$

where

$$s(\mu)dX = dx$$

The highest derivative was lost in the outer problem therefore $d^2 F/dX^2$ must be kept for the inner problem. This implies

$$s(\mu)/\mu \neq \infty \quad \text{as} \quad \mu \rightarrow 0$$

but

$$s(\mu)/\mu = 0 \quad \text{as} \quad \mu \rightarrow 0$$

implies that the solution satisfying the inner boundary condition is a multiple of X and cannot be matched. Therefore, the only possibility is that $s(\mu)/\mu = \text{a constant}$ as $\mu \rightarrow 0$, and this constant is assumed to be 1.

This gives

$$\frac{d^2 F}{dX^2} + \frac{dF}{dX} = b\mu \quad (6)$$

with

$$F(0) = 0, \quad F(1/\mu) = 1, \quad s(\mu) = \mu$$

In the inner variables $X = x/\mu$ and F , the following expansions are produced:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots$$

$$\frac{dF}{dX} = \frac{dF_0}{dX} + \mu \frac{dF_1}{dX} + \mu^2 \frac{dF_2}{dX} + \dots$$

$$\frac{d^2F}{dX^2} = \frac{d^2F_0}{dX^2} + \mu \frac{d^2F_1}{dX^2} + \mu^2 \frac{d^2F_2}{dX^2} + \dots$$

which when substituted into 5 yield

$$\frac{d^2F_0}{dX^2} + \frac{dF_0}{dX} = 0 \quad \text{and} \quad F(0) = 0$$

and finally,

$$F_0(X) = A(1 - e^{-X}) \quad (7)$$

where the outer boundary condition has been discarded.

To carry out the matching,

$$\text{1-term outer expansion} = (1-b) + bX$$

$$\text{in inner variables} = (1-b) + b\mu X$$

$$\text{expanded for small } \mu = (1-b) + b\mu X$$

$$\text{1 term inner expansion} = (1-b)$$

$$\text{1-term inner expansion} = A(1 - e^{-X})$$

$$\text{in outer variables} = A(1 - e^{-X/\mu})$$

$$\text{expanded for small } \mu = A$$

$$\text{1 term outer expansion} = A$$

matching first term outer to first term inner gives

$$A = 1-b$$

therefore

$$F_0 = (1-b)(1 - e^{-X})$$

so that now

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots$$

$$F = (1-b)(1-e^{-X}) + \text{Terms}$$

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \dots$$

$$f = (1-b) + bX + \text{NO MORE TERMS}$$

to find the second term inner expansion

$$\frac{d^2 F_1}{dX^2} + \frac{dF_1}{dX} = b$$

$$F_1 = c(e^{-X} + 1) + bX$$

therefore

$$F = (1-b)(1-e^{-X}) + \mu [c(e^{-X} + 1) + bX] + \dots \quad (8)$$

Since the exact solution for the outer variable is known, the matching from 1 to 2 may be skipped as it simply yields,

$$1-b = 1-b$$

Going now from $m = 2$ and $n = 2$,

$$\text{2-term outer expansion} = (1-b) + bX$$

$$\text{rewritten in inner variables} = (1-b) + \mu bX$$

$$\text{expanded for small } \mu = (1-b) + \mu bX$$

$$\text{2 term inner expansion} = (1-b) + \mu bX$$

$$\text{2-term inner expansion} = (1-b)(1-e^{-X}) + \mu [c(e^{-X} - 1) + bX]$$

$$\text{in outer variables} = (1-bX)(1-e^{-X}) + \mu [c(e^{-X} - 1) + bX]$$

$$\text{expanded for small } \mu = (1-b) - \mu c + bX + \dots$$

$$\text{2 term outer expansion} = (1-b) + bX - \mu c$$

$$\text{rewrite in inner variables} = (1-b) + \mu bX - \mu c$$

matching these gives

$$bx = bx - c$$

$$c = 0$$

therefore

$$f = (1-b) + b x$$

$$F = (1-b)(1 - e^{-x}) + \mu b x \quad (9)$$

This form of solution is necessary when no single asymptotic expansion is uniformly valid throughout the field of interest in a problem. This problem may arise through factors such as two characteristic lengths as those of the boundary layer thickness near the body compared to the chord length of the body.

This long sample solution is presented above so that as the solution of the earth-moon problem is discussed an understanding of the procedures involved can be felt even though the actual solution is too lengthy to be presented in detail. The procedures are the same in the orbit problem as in the example just discussed but more complicated to carry out.

B. NATURE OF THE PROBLEM

With the method of inner and outer expansions previewed, attention is now turned to the trajectory problem in earth-moon space. It is beneficial to observe first the general problem which must be studied, the three-body problem.

The three-body problem is one aspect of the n-body problem of mechanics. In earth-moon space the problem may

be depicted as in Figure 3. This figure represents the restricted three-body problem which is the second simplest of the many-body problems. The restricted three-body problem, however, is still unsolvable in a closed analytic manner. The only n-body problem which has been solved is the two-body problem. It is therefore of interest to observe the nature of the three-body problem and determine what information one may find from determinable quantities.

Using the system of three masses depicted in Figure 4 it is possible to define three position vectors in space by \bar{r}_1 , \bar{r}_2 , and \bar{r}_3 . Now, if the conventional methods of vector mechanics are applied to the system and the necessary algebraic manipulations are carried out it is possible to determine ten of the necessary eighteen integrals of the total solution to the three-body problem.

Writing the equations of motion with reference to Figure 4

$$\begin{aligned}
 m_1 \ddot{\bar{r}}_1 &= G m_1 m_2 \frac{\bar{r}_{12}}{r_{12}^3} + G m_1 m_3 \frac{\bar{r}_{13}}{r_{13}^3} \\
 m_2 \ddot{\bar{r}}_2 &= G m_1 m_2 \frac{\bar{r}_{21}}{r_{21}^3} + G m_2 m_3 \frac{\bar{r}_{23}}{r_{23}^3} \\
 m_3 \ddot{\bar{r}}_3 &= G m_1 m_3 \frac{\bar{r}_{31}}{r_{31}^3} + G m_3 m_2 \frac{\bar{r}_{32}}{r_{32}^3}
 \end{aligned} \tag{10}$$

using the fact that

$$\bar{r}_j = -\bar{r}_{ji} \quad \text{and} \quad r_{ij}^2 = r_{ji}^2$$

and adding the three equations

$$m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 + m_3 \ddot{\vec{r}}_3 = G \left(m_1 m_2 \frac{\vec{r}_{12}}{r_{12}^3} + m_1 m_3 \frac{\vec{r}_{13}}{r_{13}^3} + m_2 m_1 \left(-\frac{\vec{r}_{12}}{r_{12}^3} \right) \right. \\ \left. + m_2 m_3 \frac{\vec{r}_{23}}{r_{23}^3} + m_3 m_1 \left(-\frac{\vec{r}_{13}}{r_{13}^3} \right) + m_3 m_2 \left(-\frac{\vec{r}_{23}}{r_{23}^3} \right) \right)$$

which yields

$$\sum m_i \ddot{\vec{r}}_i = 0 \quad (11)$$

which is the conservation of linear momentum.

This vector equation integrated twice then gives six of the ten available integrals.

Now using the relation

$$\vec{r}_i \times \vec{r}_{ij} = \vec{r}_i \times \vec{r}_j - \vec{r}_i \times \vec{r}_i = \vec{r}_i \times \vec{r}_j$$

on equations 10 the following equations are obtained:

$$m_1 (\vec{r}_1 \times \ddot{\vec{r}}_1) = G \frac{m_1 m_2}{r_{12}^3} (\vec{r}_1 \times \vec{r}_2) + G \frac{m_1 m_3}{r_{13}^3} (\vec{r}_1 \times \vec{r}_3)$$

$$m_2 (\vec{r}_2 \times \ddot{\vec{r}}_2) = G \frac{m_2 m_1}{r_{21}^3} (\vec{r}_2 \times \vec{r}_1) + G \frac{m_2 m_3}{r_{23}^3} (\vec{r}_2 \times \vec{r}_3)$$

$$m_3 (\vec{r}_3 \times \ddot{\vec{r}}_3) = G \frac{m_3 m_1}{r_{31}^3} (\vec{r}_3 \times \vec{r}_1) + G \frac{m_3 m_2}{r_{32}^3} (\vec{r}_3 \times \vec{r}_2)$$

If the vector relationship

$$\vec{r}_i \times \vec{r}_j = -\vec{r}_j \times \vec{r}_i$$

is applied and the equations are added the following equation is found:

$$\sum_i m_i (\vec{r}_i \times \ddot{\vec{r}}_i) = G \left\{ \frac{m_1 m_2}{r_{12}^3} \left[(\vec{r}_1 \times \vec{r}_2) - (\vec{r}_1 \times \vec{r}_2) \right] + \text{etc} \right\}$$

which reduces to

$$\sum m_i (\vec{r}_i \times \ddot{\vec{r}}_i) = 0$$

noting that

$$\frac{d}{dt} (\vec{r} \times \dot{\vec{r}}) = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \ddot{\vec{r}}$$

the equation

$$\sum m_i (\vec{r}_i \times \ddot{\vec{r}}_i) = 0$$

may be rewritten as

$$\int m_i [\dot{\vec{r}}_i \times \dot{\vec{r}}_i] dt = m_i [\vec{r}_i \times \dot{\vec{r}}_i]$$

which finally yields

$$\sum m_i [\vec{r}_i \times \dot{\vec{r}}_i] = \text{constant} \quad (12)$$

which is conservation of angular momentum.

This allows the determination of three more of the necessary integrals for the three-body problem.

The following series of algebraic equations will lead to the determination of the final integral for the three-body problem. Writing Newton's law and operating on it one obtains:

$$\vec{F}_i = m_i \ddot{\vec{r}}_i$$

$$\vec{F}_i \cdot \frac{d\vec{r}_i}{dt} = m_i (\dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i)$$

$$\int \vec{F}_i \cdot d\vec{r}_i = \frac{m_i \dot{\vec{r}}_i^2}{2}$$

$$\frac{1}{2} m_i v_i^2 = \int \vec{F}_i \cdot d\vec{r}_i$$

Using this operation on Equation 10

$$\frac{1}{2} m_1 v_1^2 = G m_1 m_2 \int \frac{\bar{r}_{12} \cdot d\bar{r}_1}{r_{12}^3} + G m_1 m_3 \int \frac{\bar{r}_{13} \cdot d\bar{r}_1}{r_{13}^3}$$

$$\frac{1}{2} m_2 v_2^2 = G m_2 m_1 \int \frac{\bar{r}_{21} \cdot d\bar{r}_2}{r_{21}^3} + G m_2 m_3 \int \frac{\bar{r}_{23} \cdot d\bar{r}_2}{r_{23}^3}$$

$$\frac{1}{2} m_3 v_3^2 = G m_3 m_1 \int \frac{\bar{r}_{31} \cdot d\bar{r}_3}{r_{31}^3} + G m_3 m_2 \int \frac{\bar{r}_{32} \cdot d\bar{r}_3}{r_{32}^3}$$

the equations are now added to find

$$\sum \frac{1}{2} m_i v_i^2 = G m_i m_j \int \frac{\bar{r}_{ij} \cdot d\bar{r}_i}{r_{ij}^3} + G m_i m_j \int \frac{\bar{r}_{ij} \cdot d\bar{r}_j}{r_{ij}^3}$$

$$= - G m_i m_j \int \frac{d\bar{r}_{ij}}{r_{ij}^2}$$

$$= - G \frac{m_i m_j}{r_{ij}}$$

so that

$$\sum \left(\frac{1}{2} m_i v_i^2 - G \frac{m_i m_j}{r_{ij}} \right) = \text{CONST} \quad (13)$$

This is the tenth integral which represents the total conservation of energy. With eighteen integrals necessary for solution and only ten integrals that are defineable further pursuit of an analytic solution of the three-body problem is impossible.

If a restricted three-body system is assumed one further integral may be found to offer new information. This

integral, called the Jacobi integral, does not solve the problem in analytic form however.

The following two assumptions are adopted to formulate a restricted three-body problem:

1) $m_3 \ll m_1$ and m_2

2) m_1 and m_2 are in circular orbits about each other.

With this preliminary introduction to the three-body and the restricted three-body problems concluded attention will now be turned to a singular perturbation solution to the restricted three-body problem formulated by Lagerstrom and Kevorkian [2]. The development will be outlined in notation of Reference 2, however for more detailed discussion and mathematical details the reader is referred to Ref. 2.

C. FORMULATION OF THE RESTRICTED THREE-BODY PROBLEM

With a brief discussion of the general three-body problem completed and with an example problem of matched inner and outer expansions discussed, attention is now turned toward a discussion of Lagerstrom and Kevorkian's work in the application of this expansion solution to a restricted three-body problem in earth-moon space.

The planar restricted three-body problem may be formulated considering the notation used in Figure 5. In this figure $x-y$ is an earth centered frame of reference and $\bar{x}-\bar{y}$ is a moon centered frame of reference.

Motion in the x-y system may be considered to be Keplerian about the earth with only small perturbation terms due to the moon until the particle comes into the influence of the moon's gravity field. At this time the perturbation from the moon is no longer small, hence the second characteristic length which results in the singular perturbation problem and allows the use of the boundary layer "stretching" type solutions. The effect of the moon destroys the uniform validity of the Keplerian solution about the earth. If the mass fraction of the earth-moon system is used as a scale factor one may discuss the motion near the moon in terms of the inner or $\bar{x}-\bar{y}$ stretched coordinate system.

Considering the outer problem first it is possible to write the planar equations of motion in x-y space directly as;

$$\begin{aligned} \frac{d^2x}{dt^2} + (1-\mu)\frac{x}{r^3} &= \mu f \\ \frac{d^2y}{dt^2} + (1-\mu)\frac{y}{r^3} &= \mu g \end{aligned} \quad (14)$$

where

$$f = \frac{x_m - x}{r_m^3} - \xi_m \quad \text{and} \quad g = \frac{y_m - y}{r_m^3} - \eta_m$$

If the terms on the right hand side of the previous equations are set to zero, one has the Keplerian or two-body equations near the earth. The Keplerian integrals will change little near the earth. These integrals are those

of energy, angular momentum, and position. Recalling the equations for the three-body problem discussed earlier one has,

$$\sum m_i (\vec{r}_i \times \vec{v}_i) = \text{CONST}$$

and

$$\sum \left(\frac{1}{2} m_i v_i^2 - \frac{G m_i m_j}{r_{ij}} \right) = \text{CONST}$$

Rewriting these in two-dimensional coordinates in the notation of Lagerstrom one has

$$\frac{1}{2} [\dot{x}^2 + \dot{y}^2] - \frac{1-\mu}{r} = \text{CONST} = h_e \quad (15)$$

and

$$x\dot{y} - y\dot{x} = \text{CONST} = l_e \quad (16)$$

Two equations specifying the position of the orbit in the x-y plane may also be written as

$$\begin{aligned} p_e &= l_e \dot{x} + (1-\mu)y/r \\ q_e &= -l_e \dot{y} + (1-\mu)x/r \end{aligned} \quad (17)$$

These are written such that if $p_e = 0$ the motion is symmetric about the x-axis. Also if $q_e > 0$ the x-coordinate of the perigee is negative for $l_e \neq 0$ and always non-negative for $l_e = 0$.

If the initial conditions are specified in terms of these integrals, (which will be done shortly) the motion of the particle in the Keplerian orbit may be determined.

The launch is considered to be at perigee, hence $p_e = 0$ is given.

The angular momentum is defined also to be $rv \cos \phi$. If perigee is considered $\phi = 0$ and $\cos \phi = 1$, therefore, $\lambda_e = rv$. If this is non-dimensionalized with respect to the angular momentum of the earth-moon system (moon's period) the non-dimensional angular momentum may be defined as,

$$\lambda = \frac{rv}{[\mu D G (m_e + m_m)]^{1/2}} \quad (18)$$

The non-dimensional energy is defined in terms of a non-dimensional potential energy and a non-dimensional kinetic energy as,

$$\rho^2 = \left[(1-\mu) \frac{D}{r} - v^2 D / 2 G (m_e + m_m) \right] \quad (19)$$

Since h_e , λ_e , p_e determine q_e the particle initial conditions are specified in terms of h_e , p_e , λ_e , and q_e . Writing these in their non-dimensional form,

$$h_e = -\rho^2, \quad \lambda_e = \mu^{1/2} \lambda, \quad p_e = 0, \quad q_e > 0 \quad \text{and} \quad t = 0 (\mu^{3/2})$$

one is now able to specify the initial conditions of velocity, altitude, and position in terms of initial Keplerian integrals. The following Keplerian relationships at perigee are known:

$$\begin{aligned} -x_p &= r_p = a(1-e) \\ \dot{x} &= 0 \\ \dot{y}^2 &= GM \left(\frac{2}{r} - \frac{1}{a} \right) \end{aligned}$$

Replacing h_e by $-\rho^2$, λ_e by $\mu^{1/2} \lambda$ and p_e by 0 at $x = 0$ and carrying out some algebraic manipulations the initial

values for the coordinates and velocities become:

$$x = -\frac{1-\mu}{2\rho^2} \left\{ 1 - \left[1 - \frac{2\mu\rho^2\lambda^2}{(1-\mu)^2} \right]^{1/2} \right\}$$

$$y = 0 \quad (20)$$

$$\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{2\mu^{1/2}\lambda\rho^2}{1-\mu} \left[1 - \left(1 - \frac{2\mu\rho^2\lambda^2}{(1-\mu)^2} \right)^{1/2} \right]$$

where

$$a = (1-\mu)/2\rho^2$$

and

$$e = \left\{ 1 - \left[2\mu\rho^2\lambda^2/(1-\mu)^2 \right] \right\}^{1/2}$$

It is also beneficial to transform the motion equations to the independent variable x rather than t . One transformation is shown below and the rest follow similar steps to produce the transformed equations. Writing

$$\dot{y} = \frac{dy}{dx} \dot{x} = \frac{dy/dx}{dt/dx}$$

$$\frac{d}{dt}(\dot{y}) = \frac{d\dot{y}}{dx} \frac{dx}{dt} = \frac{d}{dx} \left[\frac{dy/dx}{dt/dx} \right] \frac{1}{t'}$$

$$= \frac{d}{dx} \left[\frac{y'}{t'} \right] \frac{1}{t'}$$

$$= \frac{y''t' - y't''}{(t')^2} \frac{1}{t'}$$

finally

$$\frac{d^2y}{dt^2} = \frac{y''t'}{(t')^2} - \frac{y't''}{(t')^3}$$

A similar development may be used to transform d^2x/dt^2 , h_e , and l_e so that these equations are finally available in the form:

$$-\frac{t''}{t'^3} + (1-\mu) \frac{y}{r^3} = \mu f$$

$$\frac{y''}{t'^2} - \frac{t'' y'}{t'^3} + (1-\mu) \frac{y}{r^3} = \mu g \quad (21)$$

$$\frac{d h_e}{d x} = \frac{d}{d x} \left[\frac{1+y^2}{2t'^2} - \frac{1-\mu}{r} \right] = \mu [f + g y']$$

$$\frac{d l_e}{d x} = \frac{d}{d x} \left[\frac{x y' - y}{t'} \right] = \mu t' (x g - y f)$$

D. OUTER EXPANSION

Attempting a solution of the equations of motion for the particle in earth-moon space in a power series in μ one has;

$$y(x, \mu) = \mu^{3/2} y_{1/2}(x, \mu) + \mu y_1(x) + \dots \quad (22)$$

$$t(x, \mu) = t_0(x, \mu) + \mu t_1(x) + \dots \quad (23)$$

The terms $\mu^{3/2} y_{1/2}$ and t_0 are the exact solutions for the equations of motion with the perturbation terms on the right hand side set to zero. In common with celestial mechanics notation, these terms are written as,

$$t_0 = a^{3/2} [E(x) - e \sin E(x)] \quad (24)$$

$$\mu^{3/2} y_{1/2} = -b \sin E(x) \quad (25)$$

where

$$E(x) = \sin^{-1} \left[1 - (e - \frac{x}{a})^2 \right]^{1/2}$$

$$b = a(1-e^2)^{1/2}$$

$$a = (1-\mu)/2\rho^2$$

$$e = [1 - 2\mu\rho^2\lambda^2/(1-\mu)^2]^{1/2}$$

If $t_0(x, \mu)$ and $\mu^{1/2} y_{1/2}(x, \mu)$ are further expanded as

$$t_0(x, \mu) = t_{00}(x) + \mu t_{01}(x) + \dots \quad (26)$$

$$y_{1/2}(x, \mu) = y_{1/20}(x) + \mu y_{1/21}(x) + \dots \quad (27)$$

where by appropriate expansions in powers of μ one finds,

$$t_{00}(x) = [\sin^{-1} z \rho (x - \rho^2 x^2)^{1/2} - z \rho (x - \rho^2 x^2)^{1/2}] / z^{3/2} \rho^3 \quad (28)$$

$$t_{01}(x) = -t_{00}(x) + [2(1 - \rho^2 \lambda^2)x^2 + 3x\lambda^2] / z[2x(1 - \rho^2 x)]^{1/2} \quad (29)$$

$$y_{1/20}(x) = z^{1/2} \lambda (x - \rho^2 x^2)^{1/2} \quad (30)$$

$$y_{1/21}(x) = -\lambda \rho (2x + \lambda^2)(1 - 2\rho^2 x) / z \rho [2x(1 - \rho^2 x)]^{1/2} \quad (31)$$

It is also necessary to insure that the moon and the particle pass within the scale distance μ of each other. The phase constant expansion which includes this condition in the analysis is,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 \quad (32)$$

where

$$T_0 = t_{00}(1)$$

$$T_{1/2} = -y_{1/20}(1)$$

and T_1 is an arbitrary term which assures that the moon may be allowed to interact with the particle as desired.

The perturbation terms $t_1(x)$ and $y_1(x)$ may also be determined, the details of which are long and tedious and may be found on pages 689-690 of Shi and Eckstein [6]. The final form of these equations is:

$$t_1(x) = \int_0^x f_0(z) [\alpha(x) - \alpha(z)] dz \quad (33)$$

$$y_1(x) = x \int_0^x z t'_{00}(z) g_0(z) [\beta(x) - \beta(z)] dz \quad (34)$$

where

$$\alpha(z) = -[z^{1/2}(1-\rho^2 z)^{1/2}/2 + z^{3/2}/(1-\rho^2 z)^{1/2} - (\frac{3}{2}\rho) S_{1,1} \rho z^{1/2}] / 2^{1/2} \rho^3 \quad (35)$$

and

$$\beta(z) = - \left\{ \left[\frac{1}{z(1-\rho^2 z)} \right]^{1/2} \right\} \quad (36)$$

also $f_0(z)$ and $g_0(z)$ are given in terms of the moon's position coordinates as,

$$f_0(z) = \frac{\cos[t_{00}(z) - T_0] - z}{\{1 + z^2 - 2z \cos[t_{00}(z) - T_0]\}^{1/2}} - \cos[t_{00}(z) - T_0] \quad (37)$$

$$g_0(z) = \frac{S_{1,1} [t_{00}(z) - T_0] - z}{\{1 + z^2 - 2z \cos[t_{00}(z) - T_0]\}^{3/2}} - S_{1,1} [t_{00}(z) - T_0] \quad (38)$$

If $t_1(x)$ and $y_1(x)$ are now expanded asymptotically as $x \rightarrow 1$ the following expansions are produced:

$$t_1(x) = (3-2\rho^2)^{-3/2} \log(1-x) - \gamma(\rho) + o(1) \quad (39)$$

$$y_1(x) = (3-2\rho^2)^{-3/2} \log(1-x) + \delta(\rho) + o(1) \quad (40)$$

where

$$\gamma = \left\{ \int_0^1 f_0(z) [\alpha(1) - \alpha(z) + \frac{1}{Q(1-z)}] dz + \frac{1}{Q} \right\} \quad (41)$$

$$\delta = \int_0^1 \left\{ z t'_{00}(z) q_0(z) [\beta(1) - \beta(z)] + \frac{1}{Q(1-z)} \right\} dz + \frac{1}{Q} \quad (42)$$

with

$$Q = (1 + U^2)^{3/2} \quad (43)$$

and

$$U = 1/t'_{00}(1) \quad (44)$$

With the terms t_0 , t_1 , y_1 , and $\mu^{1/2} y_{1/2}$ now available the outer expansion for t and y is known and is ready to be matched near the moon.

E. INNER EXPANSION

As the vehicle approaches the moon it comes into a hyperbolic encounter with this body. The hyperbolic encounter may be expressed in terms of inner variables. These inner variables are written as,

$$\bar{x} = \frac{x - \cos(t-T)}{\mu}, \quad \bar{y} = \frac{y - \sin(t-T)}{\mu}, \quad \bar{t} = \frac{t - T_0 - \mu \bar{z}}{\mu}$$

where μ is the scale factor for the inner variables and τ is a constant which must be determined.

The inner variable equations of motion are written as:

$$\frac{d^2 \bar{x}}{d\bar{t}^2} + \frac{\bar{x}}{\bar{r}^3} = - \frac{\mu(1-\mu)(\mu \bar{x} + \cos \psi)}{\bar{r}_e^2} + \mu(1-\mu) \cos \psi \quad (45)$$

$$\frac{d^2 \bar{y}}{d\bar{t}^2} + \frac{\bar{y}}{\bar{r}^3} = - \frac{\mu(1-\mu)(\mu \bar{y} + \sin \psi)}{\bar{r}_e^2} + \mu(1-\mu) \sin \psi \quad (46)$$

where

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2$$

$$\bar{r}e^2 = 1 + 2\mu(\bar{x} \cos \psi + \bar{y} \sin \psi) + \mu^2 \bar{r}^2$$

$$\psi = \mu(\bar{t} - \tau) + T_0 - T$$

In the hyperbolic encounter one may express the Keplerian solution in parametric form as,

$$\bar{x} = \bar{a}(\bar{e} - \cosh u) \cos \bar{\theta} \mp \bar{a}(\bar{e}^2 - 1)^{1/2} \sinh u \sin \bar{\theta} \quad (47)$$

$$\bar{y} = \bar{a}(\bar{e} - \cosh u) \sin \bar{\theta} \pm \bar{a}(\bar{e}^2 - 1)^{1/2} \sinh u \cos \bar{\theta} \quad (48)$$

$$\bar{t} = \bar{a}^{3/2} (\bar{e} \sinh u - u) \quad (49)$$

The Keplerian integrals previously discussed expressed for inner variables are

$$\bar{h} = \frac{1}{2} \left[\left(\frac{d\bar{x}}{d\bar{t}} \right)^2 + \left(\frac{d\bar{y}}{d\bar{t}} \right)^2 \right] - \frac{1}{\bar{r}}$$

$$\bar{l} = \bar{x} \frac{d\bar{y}}{d\bar{t}} - \bar{y} \frac{d\bar{x}}{d\bar{t}} \quad (50)$$

$$\bar{p} = \bar{l} \frac{d\bar{x}}{d\bar{t}} + \bar{y}/\bar{r}$$

$$\bar{q} = -\bar{l} \frac{d\bar{y}}{d\bar{t}} + \bar{x}/\bar{r}$$

These integrals are related to the hyperbolic elements by,

$$\bar{a} = 1/2h, \quad \bar{e} = \sqrt{1 + 2h\bar{l}^2}$$

$$\bar{p} = -\bar{e} \sin \bar{\theta}, \quad \bar{q} = -\bar{e} \cos \bar{\theta}$$

To carry out the matching of the inner and outer expansions the values of \bar{y} and \bar{t} as functions of \bar{x} along the approach asymptote must be determined. These functional

values may be determined by allowing $u \rightarrow \infty$. As $u \rightarrow \infty$ the following functional forms of \bar{y} and \bar{t} are determined to be:

$$\bar{y} = \frac{V_1}{U_1} \bar{x} - \frac{AV_1 - BU_1}{U_1} + o(\bar{x}^{-1}) \quad (51)$$

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 \bar{a}^{3/2} \bar{e}} + o(\bar{x}^{-1}) \quad (52)$$

The U_1 and V_1 terms are the velocity components on the approach asymptote.

$$U_1 = \frac{2\bar{h} \bar{p} \bar{x} - (2\bar{h})^{1/2} \bar{g}}{1 + 2\bar{h} \bar{x}^2}$$

$$V_1 = \frac{-2\bar{h} \bar{g} \bar{x} + (2\bar{h})^{1/2} \bar{p}}{1 + 2\bar{h} \bar{x}^2}$$

where

$$U_1^2 + V_1^2 = 2\bar{h}$$

and A and B are,

$$A = -\bar{g}/2\bar{h} \quad \text{and} \quad B = -\bar{p}/2\bar{h}$$

If the negative \bar{y} -intercept is considered to be K_1 , that is to say $\bar{y} = -K_1$ for $x = 0$,

$$K_1 = \frac{AV_1 - BU_1}{U_1} \quad (53)$$

and

$$\bar{x} = K_1 U_1 \quad (54)$$

U_{11} and V_{11} , the velocity components along the departure hyperbola, may be written also

$$U_{11} = \frac{2\bar{h} \bar{p} \bar{x} + (2\bar{h})^{1/2} \bar{g}}{1 + 2\bar{h} \bar{x}^2}$$

$$V_{11} = \frac{-2\bar{h} \bar{p} \bar{x} + (2\bar{h})^{1/2} \bar{p}}{1 + 2\bar{h} \bar{x}^2}$$

F. MATCHING OF THE INNER AND OUTER SOLUTIONS

The matching of the inner and outer expansions must be carried out at this time. Returning to the example discussed, the principle behind this matching may be recalled. The only difference in the two problems is that in the present case, since only the first term of the moon inner variable expansion is needed, it is only necessary to rewrite the outer expansion in inner variables and then carry out the matching.

For the particle near the moon the x-coordinate is ξ_m . This occurs when $t = T_0$. If the x-coordinate is specified to be

$$\chi = 1 - \frac{\mu}{2} T_{1/2}^2$$

this implies that

$$\xi_m = 1 - \frac{\mu}{2} T_{1/2}^2$$

The phase relation was written as,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 = T_0 + \mu^{1/2} T_{1/2} + O(\mu) \quad (55)$$

but with $t = T_0$ the equation may be written as,

$$t - T = \mu^{1/2} T_{1/2} + O(\mu)$$

so that

$$\bar{\chi} = \bar{\xi} + O(1)$$

with

$$\bar{\chi} = \frac{\chi - \xi_m}{\mu} = \frac{\chi - 1 + \frac{\mu}{2} T_{1/2}^2}{\mu}$$

therefore

$$\bar{x} = \frac{x-1 + \frac{1}{2} T_{1/2}^2}{\mu}$$

If the outer expansion is written as

$$t = t_{\infty}(x) + \mu [t_{01}(x) + t_1(x)] + \dots \quad (56)$$

$$y = \mu^{1/2} y_{1/2}(x, 0) + \mu y_1(x) + \dots \quad (57)$$

and replacing

$$x \quad \text{by} \quad x = 1 + \mu (\bar{x} - T_{1/2}^2/2)$$

$$t = t_{\infty}(1) + \mu \left[\left(\bar{x} - \frac{T_{1/2}^2}{2} \right) U^{-1} + t_{01}(1) + (1+U^2)^{-3/2} \log \mu (-\bar{x}) \right. \\ \left. + Y(\rho) \right] + O(\mu)$$

$$t_{\infty}(1) = T_0 \quad \text{AND} \quad \bar{t} = \frac{t - T_0 - \mu \bar{z}}{\mu}$$

$$\bar{t} = \left(\bar{x} - \frac{T_{1/2}^2}{2} \right) U^{-1} + t_{01}(1) + (1+U^2)^{-3/2} \left\{ \log \mu + \log [-\bar{x}] \right\} \\ + Y(\rho) - \bar{z} + O(1) \quad (58)$$

also it is recalled that 52 was written

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 \bar{a}^{3/2} \bar{t}} + O(\bar{x}^{-1})$$

The term $(1+U^2)^{-3/2} \log \mu$ does not occur in the later equation. It must therefore be cancelled by some other term. Since the only arbitrary term in the inner expansion for \bar{t} is \bar{z} , a term in \bar{z} must cancel the $\log \mu$ term, hence,

$$\bar{z} = (1+U^2)^{-3/2} \log \mu + \bar{z}_1 \quad (59)$$

and

$$\bar{t} = U^{-1} \bar{x} + (1+U^2)^{-3/2} \log (-\bar{x}) + t_{01}(1) - \frac{T_{1/2}^2}{2U} + Y(\rho) - \bar{z}_1 + O(1) \quad (60)$$

\bar{y} may likewise be determined to be,

$$\bar{y} = \frac{T_{112}}{2U} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) - \frac{\bar{x}}{U} + o(1) \quad (61)$$

Comparison of 51 to 61 and 52 to 60 shows

$$U_1 = U$$

$$V_1 = -1$$

If the equations are written such that,

$$\bar{y} = \frac{V_1}{U_1} \bar{x} - K_1 + o(\bar{x}^{-1}) \quad (62)$$

and

$$\bar{y} = -K - \frac{\bar{x}}{U} + o(1) \quad (63)$$

where

$$-K = \frac{T_{112}}{2U} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) \quad (64)$$

$$K_1 = (AV_1 - BU_1)/U_1 \quad (65)$$

Then it is evident that

$$K_1 = K$$

Rewriting \bar{t} as

$$\bar{t} = \frac{\bar{x}}{U} + (1+U^2)^{-3/2} \log(-\bar{x}) + L + o(1) \quad (66)$$

where

$$L = t_{01}(1) - \frac{T_{112}}{2U} + \gamma(\rho) - \tau_1$$

and

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 \bar{a}^{3/2} \bar{e}} + o(x^{-1}) \quad (67)$$

then

$$L = -\frac{A}{U_1} + \bar{a}^{1/2} \log \frac{2}{U_1 \bar{a}^{1/2} \bar{e}}$$

which completes the equation matching process and determines the unknown constants.

These values are used in equations 50 to obtain from substitution

$$\begin{aligned} 2\bar{h} &= 1 + U^2 \\ \bar{l} &= KU \\ \bar{p} &= KU^2 + (1 + U^2)^{-1/2} \\ \bar{\delta} &= KU - U(1 + U^2)^{-1/2} \end{aligned} \quad (68)$$

The term τ_1 may also be completely determined to be

$$\tau_1 = t_{0,1} - \frac{\pi_{1,2}^2}{2U} + Y(\rho) + \frac{A}{U} - \bar{a}^{1/2} \log \frac{2}{\bar{a}^{1/2} U \bar{e}} \quad (69)$$

The total or composite expansion solution is the outer expansion plus the inner expansion minus the common part.

It may be written as;

$$y = \mu^{1/2} y_{1/2}(x, \mu) + \mu y_1(x) + \mu [q(\bar{x}) - \alpha^*(\bar{x})] + o(\mu) \quad (70)$$

$$t = t_0(x, \mu) + \mu t_1(x) + \mu [E(\bar{x}) - \beta^*(\bar{x})] + o(\mu) \quad (71)$$

where

$$\alpha^* = \frac{V_1}{U_1} \bar{x} - \bar{a} \bar{e} (V_1 \cos \bar{\theta} - U_1 \sin \bar{\theta}) / U_1 \quad (72)$$

$$\beta^* = (\bar{x} - \bar{a} \bar{e} \cos \bar{\theta}) / U_1$$

The solution to first order after moon passage may be obtained by a similar matching procedure as that used above.

Before leaving the analytic theory it is wise to note that this procedure of inner and outer expansions may be

carried out to higher orders of μ which increases the accuracy as the value of the parameter μ increases.

III. THE COMPUTER PROGRAM

The computer program used in this paper was written for the IBM 360 system at the Naval Postgraduate School Computer Facility. It has been run successfully on the OS/CMS time share remote terminal and the standard batch processor unit.

The program is a straight forward computation with the use of Scientific Subroutine Package programs from the sub-program library. These programs are used to evaluate the integrals which appear in the solution near the moon.

All computation is carried out in double-precision notation to maintain as much accuracy as possible. The integration routine is a double-precision sixteen point Gaussian quadrature from the SSP library in the Fortran IV language.

The necessary inputs to the program are the non-dimensional boundary conditions discussed in this paper. From these parameters the program calculates the necessary constants, then the integrals of the motion equations are computed for $x = 1$ (near the moon). With these values the program evaluates the motion in the vicinity of the moon and prints the necessary parameters. Control is then returned to the input mode and a new series of calculations may be carried out. If ρ becomes too large the machine proceeds to stop and ends processing.

The necessary equations, a test case, and the computer program are contained in the Appendix. A flow chart may be found in Figures 17 and 18. All equations are explained in References 7, 8, and 9 and the same notation is used as much as possible.

IV. DISCUSSION OF RESULTS

A. VERIFICATION OF COMPUTER RESULTS AGAINST THOSE OF REFERENCE 7

A computer program to solve the restricted earth-moon three-body problem was written. The first test of this program was against the data presented in Reference 7. Figures 6, 7, and 8 show the agreement of the present results with those of Ref. 7. The results for γ and δ also agree with Table 1 of Reference 7. As a final check examples one through nine of Reference 7 were tested and comparison of results was good.

No data of any form for values of ρ greater than 0.707 could be made to agree with the published data of Reference 7. The inability of the program to predict accurate results for this case of ρ greater than 0.707 was unresolved. The probable cause of the instability of the solution is due to the denominator of the $f_0(x)$ term rapidly vanishing near this point. The equation for $f_0(x)$ was rewritten in several forms, all to no avail. The equation was also expanded in a Taylor's series about 0. This expansion gave some improvement in the prediction of γ and δ , but the improvement was not sufficient to be used for numerical work.

B. PARAMETRIC STUDY IN THREE DIMENSIONAL EARTH-MOON TRAJECTORIES

The program used to predict the results of Reference 7 was modified to include three dimensional orbit consideration

for values of $i = 0(1)$. The modification still led to the same results for planar orbits and a study for inclined orbits was carried out. Cases were run to predict the effect of variations of energy, angular momentum, position of the moon and orbital inclination on the space vehicle as it approached the moon's position.

The present program predicts all of the important parameters near the moon. The effect on changes in initial conditions can be observed in the changes in these near moon parameters. One of the most important of these parameters and one that requires knowledge of the other parameters is the parameter of closest approach to the moon. This parameter has been chosen to show the effects of varying initial conditions on the moon encounter.

Figures 9 and 10 show how the initial launch conditions are related to altitude and velocity at launch from perigee. The altitude limit on perigee shown in Figure 9 indicates values of λ must be less than 3.0. Figure 10 shows the velocity at perigee. Since conventional launch vehicles cannot attain values of velocity much greater than around 40,000 kms/hr an outside limit on velocity was taken to be 50,000 kms/hr. This, then implies that λ should be greater than 1.0. For a given launch condition the values of ρ and λ can be related to the velocity and altitude at perigee. The perigee launch gives the angle of perigee to be π and time of launch to be $t = 0$. Hence, this defines the launch conditions completely.

Figure 11 shows the effect of changes in the initial energy, ρ , relative the earth on the distance of closest approach. It is apparent that as the energy is reduced (ρ becomes greater) the distance of closest approach to the moon is increased. The effect of orbit inclination on the distance of closest approach is also evident. This Figure predicts an increase in the distance as the inclination angle of the orbit is increased.

Figure 12 depicts the effects of changes in the angular momentum, λ , relative to the earth on the approach distance to the moon. As the angular momentum is increased (this corresponds to an increase in velocity and/or perigee height) the distance of closest approach to the moon is increased. The effect of the inclination of the orbit shows up in this Figure as before to indicate greater approach distances for higher values of inclination.

Figure 13 can be used to predict two effects which occur due to parameter changes. The effect of T_1 on the closest approach distance is due to the variation of the moon's position at launch of the vehicle. For positive values of T_1 the moon and the vehicle interact very closely, whereas for negative values of T_1 the moon passes far from the vehicle.

The effect of a shift on the distance of closest approach may also be seen in Figure 13. Smaller values of the apse angle allow for closer interaction with the moon.

The value for planar orbits is independent of the apse angle and therefore remains the same.

Figures 16, 17, and 18 show the effect of these same parameters on the distance of closest approach for orbits of only slight inclinations. As can be seen from these figures the effects of changes in ρ , λ , and T_1 are the same as before. As would be expected changes in the apse angle and small changes in the inclination angle from the planar case cause no significant change in the distance of closest approach as predicted in the planar case.

Figures 14 and 15 show the three-dimensional geometry of the earth orbit and the moon passage. The inclination angle, i , discussed is the angle between the earth-moon plane and the orbit plane. The apse angle, ω , is the angle from the node line, Ω , to the apse axis. Similar parameters are shown for the moon passage orbit.

V. CONCLUSIONS AND RECOMMENDATIONS

The method of uniformly valid asymptotic expansions appears to have merit as an initial analytic tool. The ability of this solution to predict values of parameters of earth-moon trajectories with such a small expenditure of computer time well justifies use of these expansion solutions. A quick parametric variational study could predict "ball-park" initial conditions which would greatly reduce the time of total solution of a trajectory problem.

Observing the results of the analysis of this paper it appears that one might conclude that the closest approach possible occurs when the launch energy is as high as possible. The angular momentum relative to the earth must be low also for this closest approach to occur. The moon's position which is fixed by T_1 and ω must be such that it is just ahead of the particle as they come near each other.

The only other parameter which must be considered is the launch inclination. The minimum value of closest approach is indicated to occur for planar trajectories.

From this discussion one then concludes that minimum distance of closest approach occurs for a planar launch at the highest value of energy and lowest value of angular momentum possible. These are attained for near earth launch with high perigee velocities.

It is recommended that further study should be conducted in the vicinity of the minimum energy solution region. This is the region of the previous mentioned instability and is very important in the study of low energy transfers where most of today's travel takes place.

It would also be interesting to release some of the restrictions on the analysis in order to apply the solution to interplanetary problems.

Kevorkian [8] has extended the analysis to higher order terms and a comparison of this result with those higher order solutions would also be of interest.

Additional work is necessary before the present analysis can be applied to a full spectrum of trajectory problems. The present work however indicates that useful information may be gained from this form of solution for the restricted three-body problem and that this solution has merit as a potential analytical tool for initial design studies.

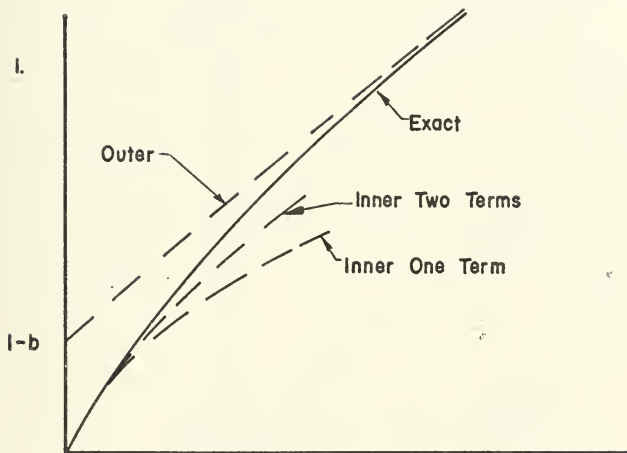


FIGURE 1. AMES, VAN DYKE BOUNDARY LAYER PROBLEM.

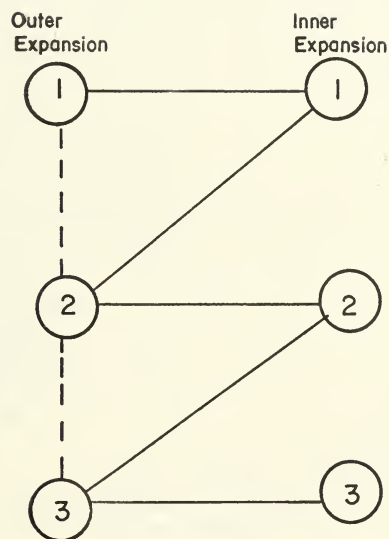


FIGURE 2. FLOW CHART OF MATCHING PRINCIPLE.

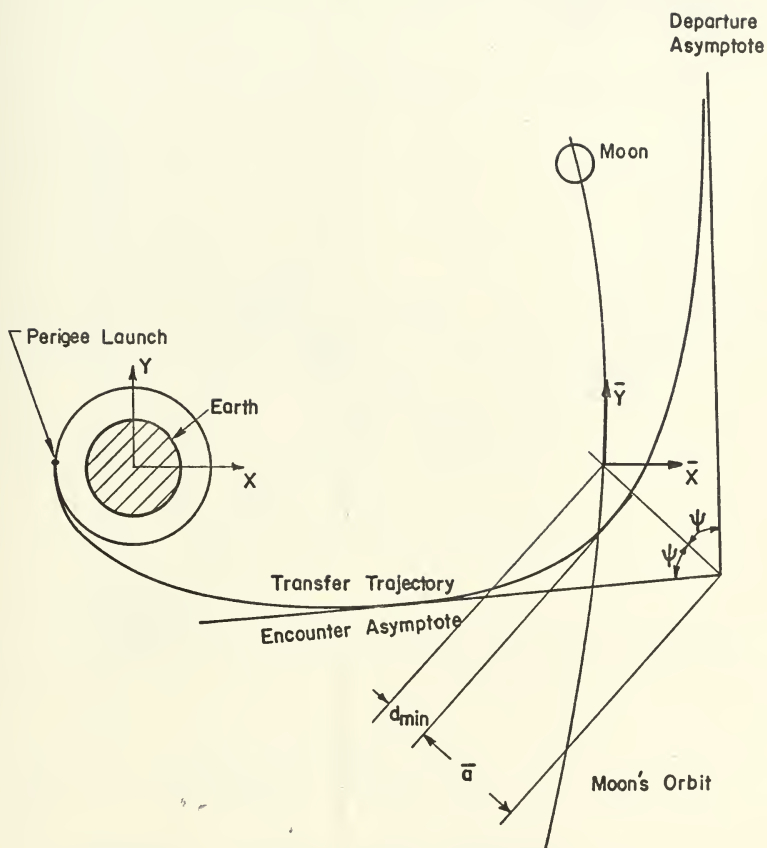


FIGURE 3. EXAMPLE EARTH-MOON HYPERBOLIC ENCOUNTER TRAJECTORY.

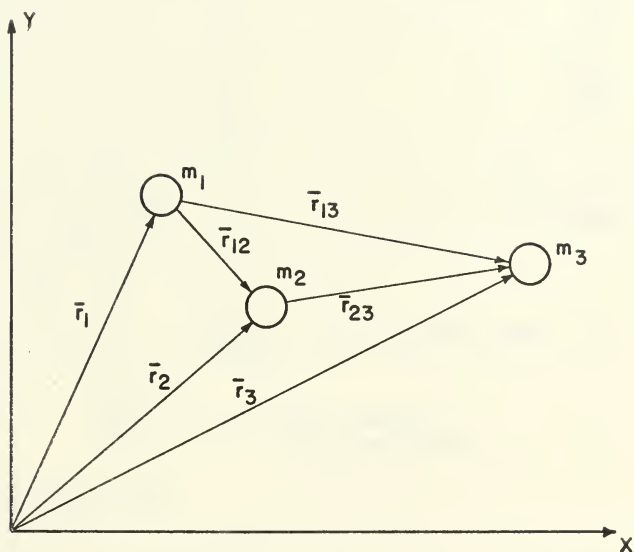


FIGURE 4, THREE - BODY GEOMETRY

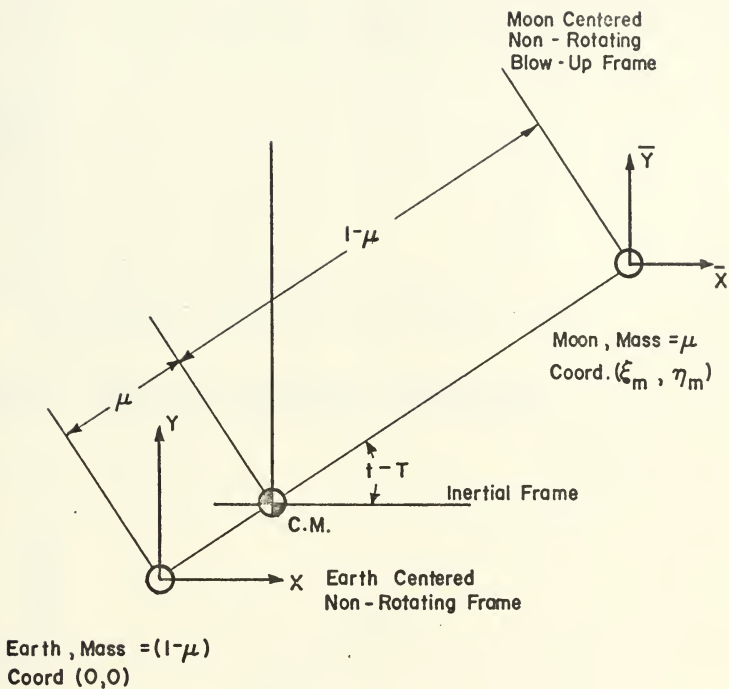


FIGURE 5 , EARTH-MOON COORDINATE SYSTEMS FOR THE RESTRICTED 3-BODY PROBLEM.

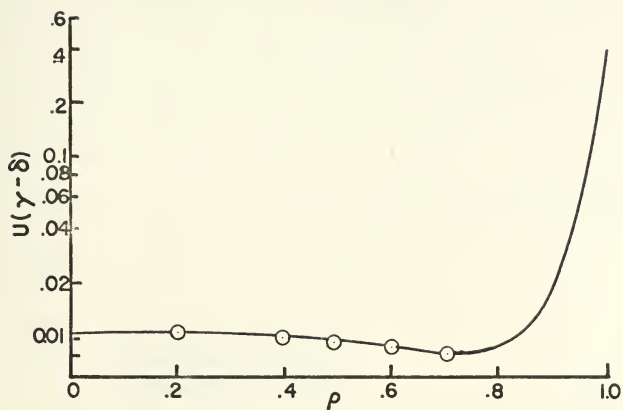


FIGURE 6. ANGULAR MOMENTUM CORRECTION.

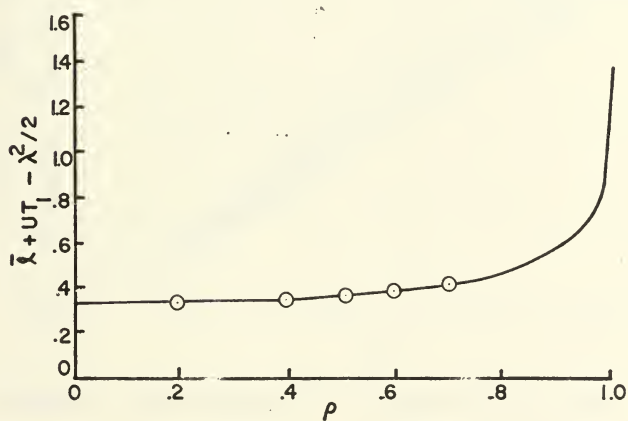


FIGURE 7. ANGULAR MOMENTUM RELATIVE TO THE MOON.

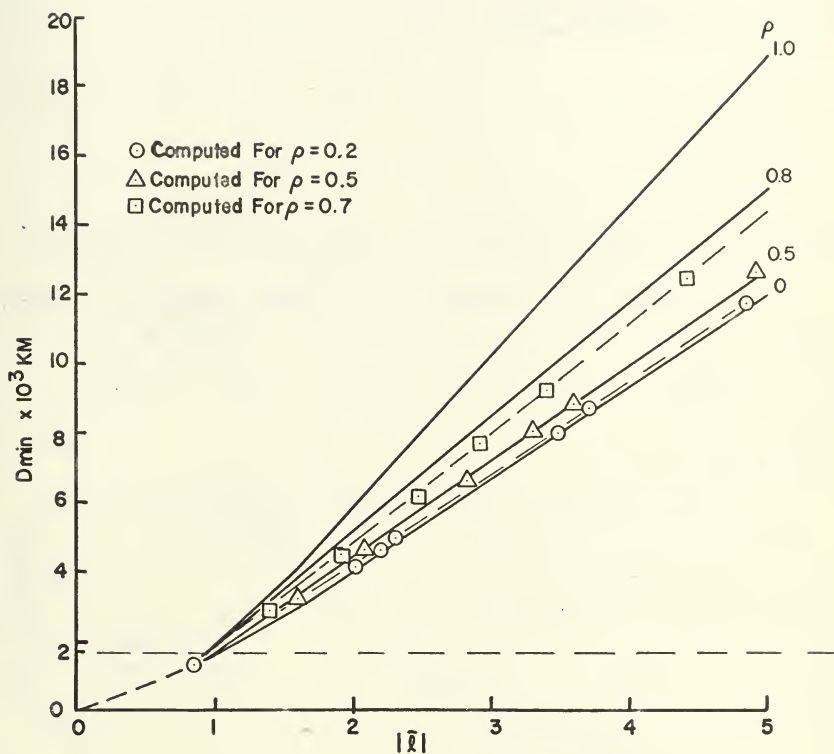


FIGURE 8. DISTANCES OF CLOSEST ENCOUNTER WITH MOON.

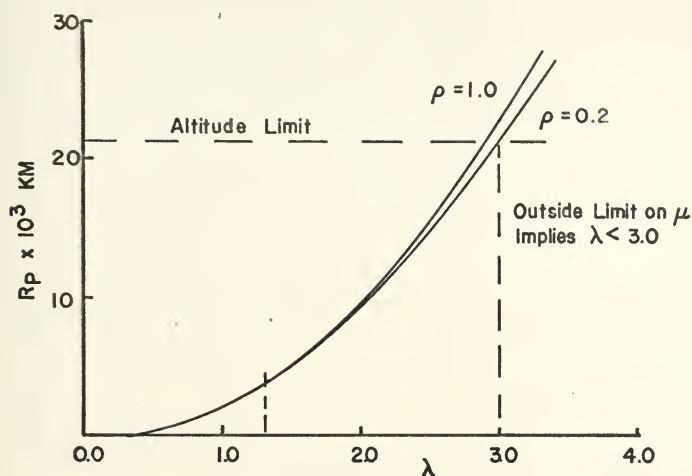


FIGURE 9. PERIGEE HEIGHT VS. LAMBDA FOR VALUES OF RHO.

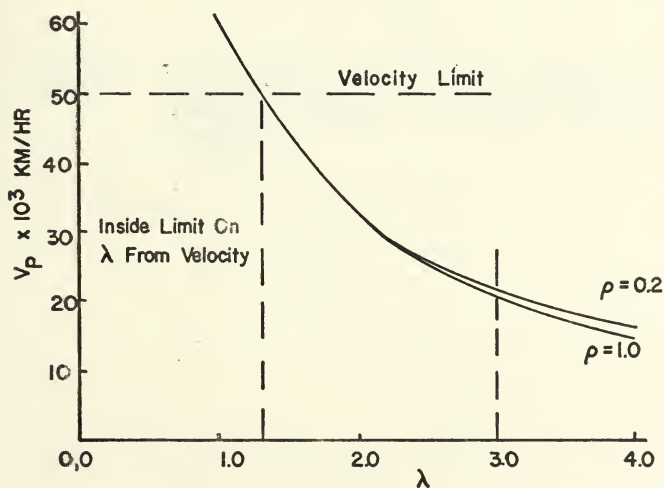


FIGURE 10. PERIGEE VELOCITY VS. LAMBDA FOR VALUES OF RHO.

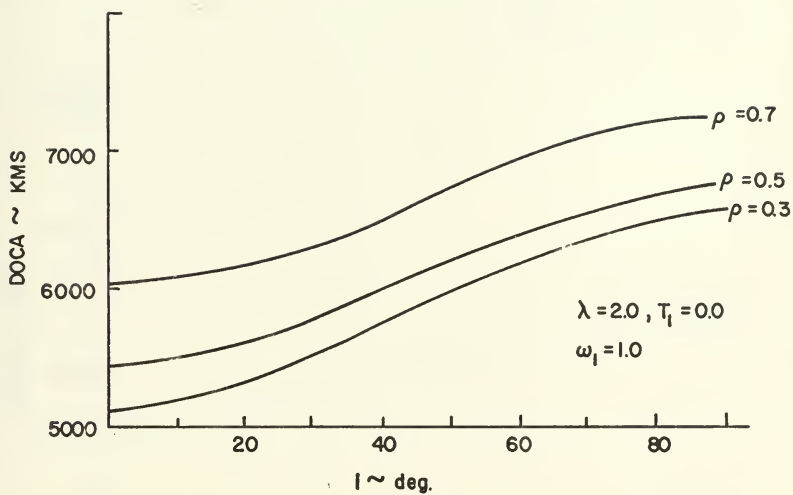


FIGURE 11. INCLINATION VS. CLOSEST APPROACH FOR VARYING ENERGY, ALL OTHER PARAMETERS CONSTANT.

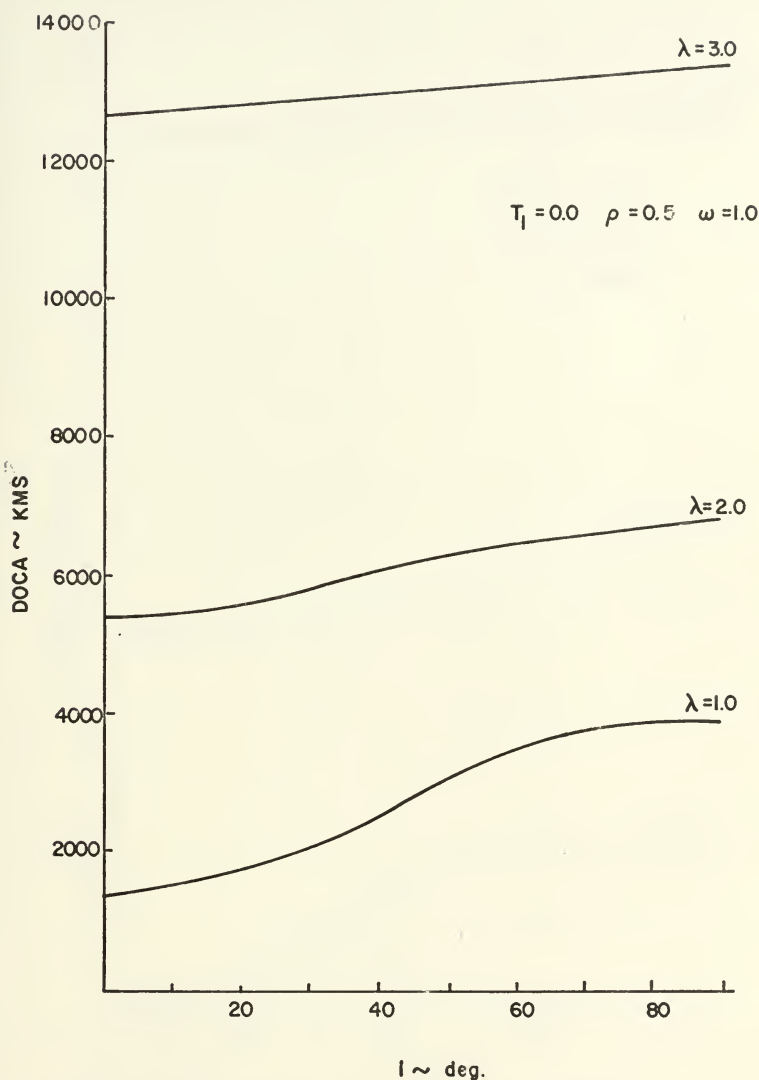


FIGURE 12. INCLINATION VS. CLOSEST APPROACH FOR VARYING ANGULAR MOMENTUM, ALL OTHER PARAMETERS CONSTANT.

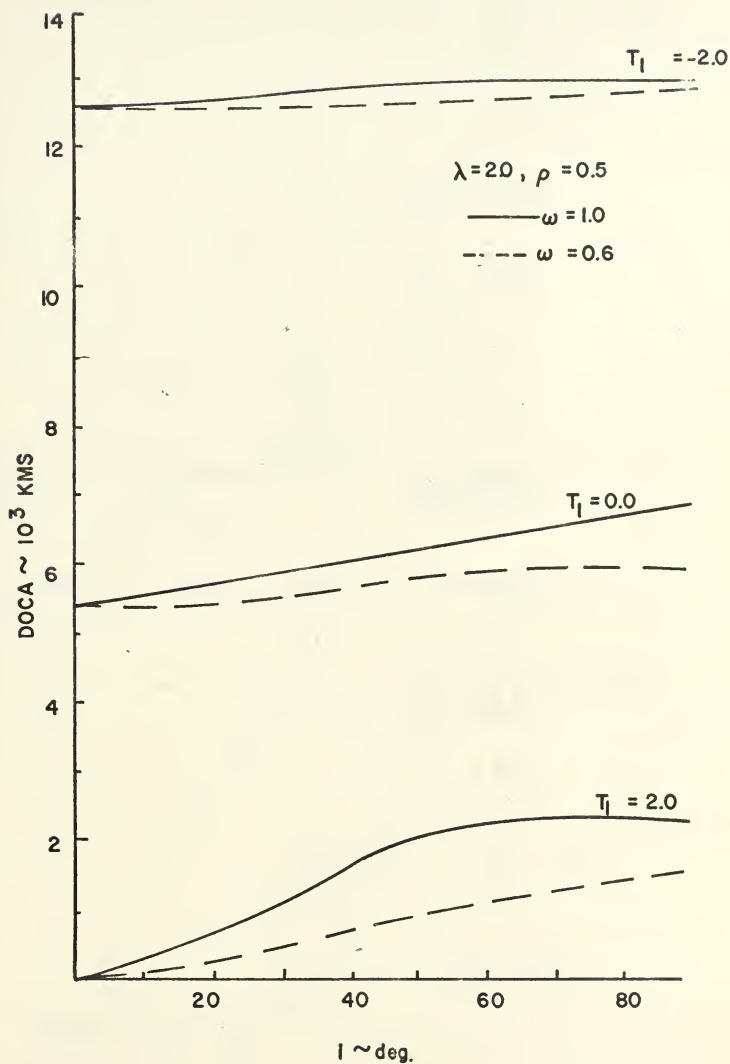


FIGURE 13. INCLINATION VS. CLOSEST APPROACH FOR VARYING PHASE CONSTANT (T_1) ALL OTHER PARAMETERS CONSTANT.

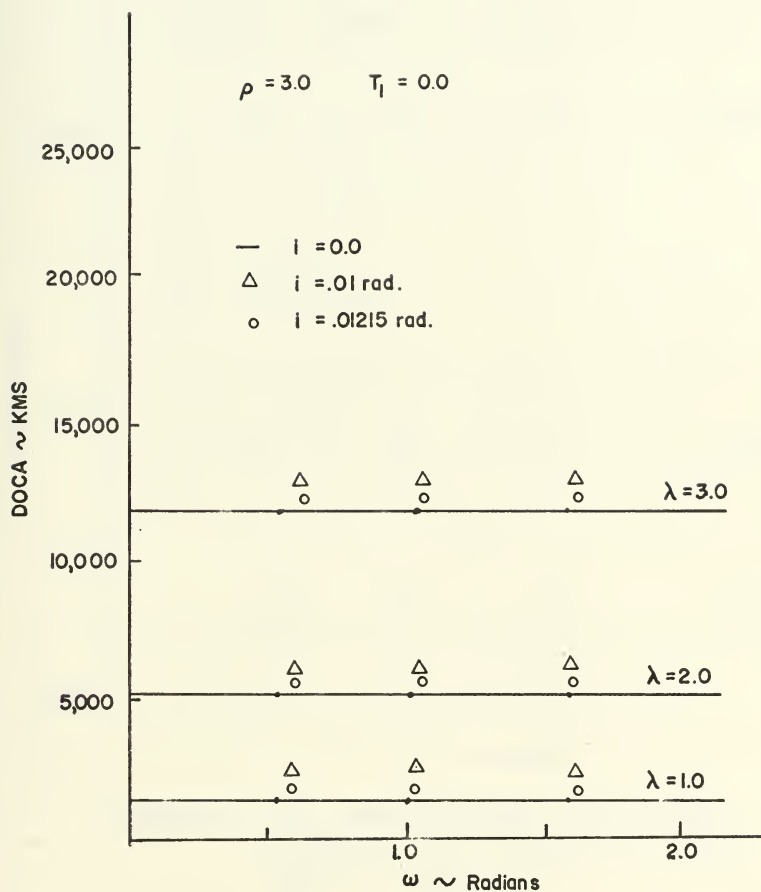


FIGURE 16. APSE ROTATION ANGLE VS. CLOSEST APPROACH
 FOR 3 VALUES OF i WITH ρ AND T_1 CONSTANT

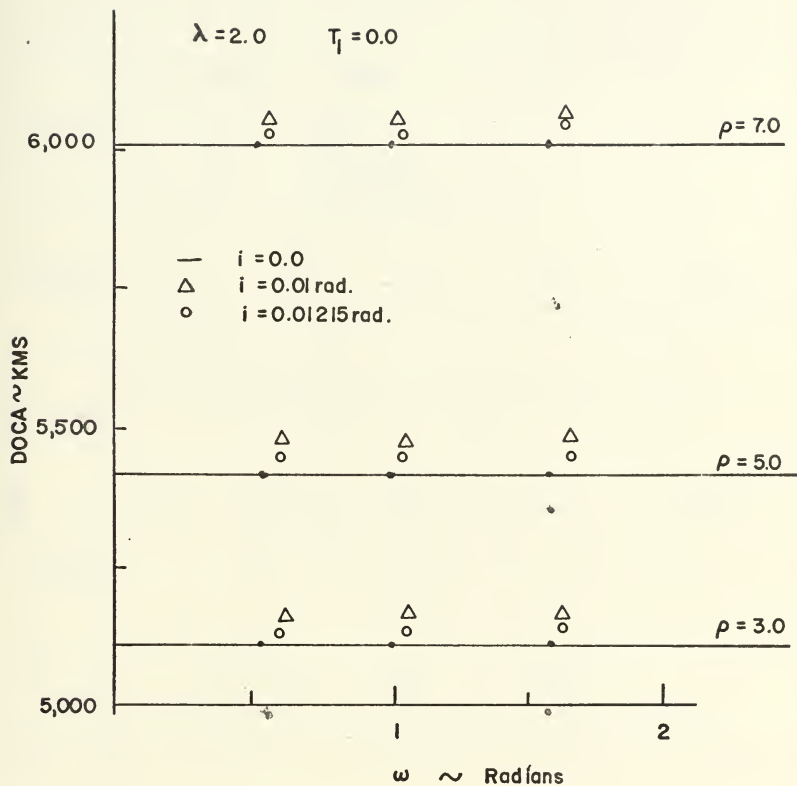


FIGURE 17. APSE ROTATION ANGLE VS. CLOSEST APPROACH FOR 3 VALUES OF i WITH λ AND T_1 FIXED

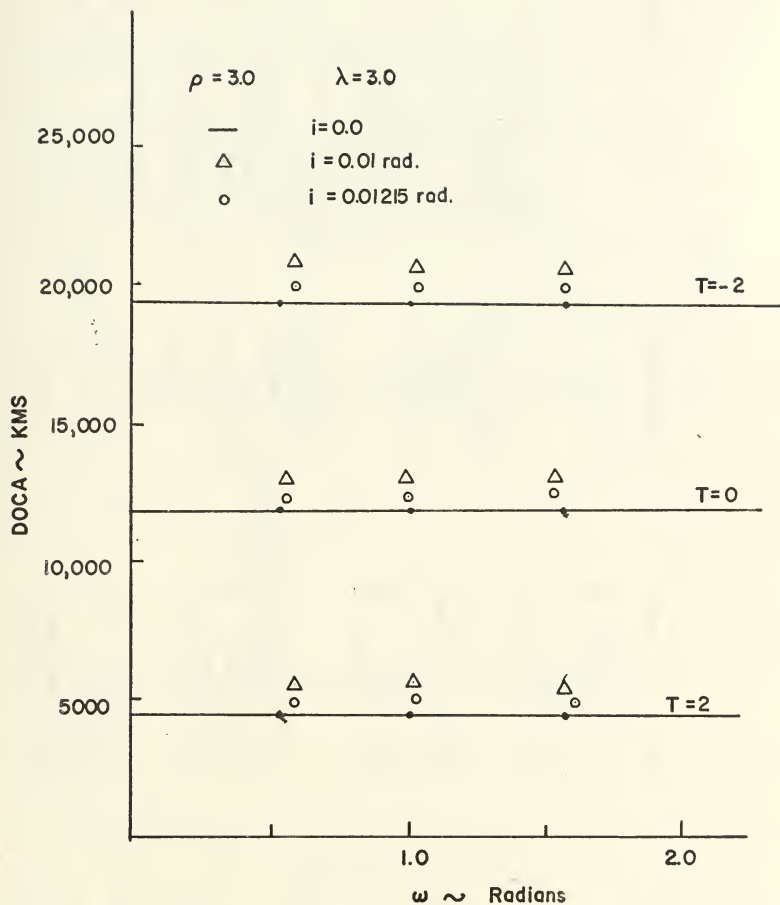


FIGURE 18. APSE ROTATION ANGLE VS. CLOSEST APPROACH FOR 3 VALUES OF i WITH ρ AND λ CONSTANT.

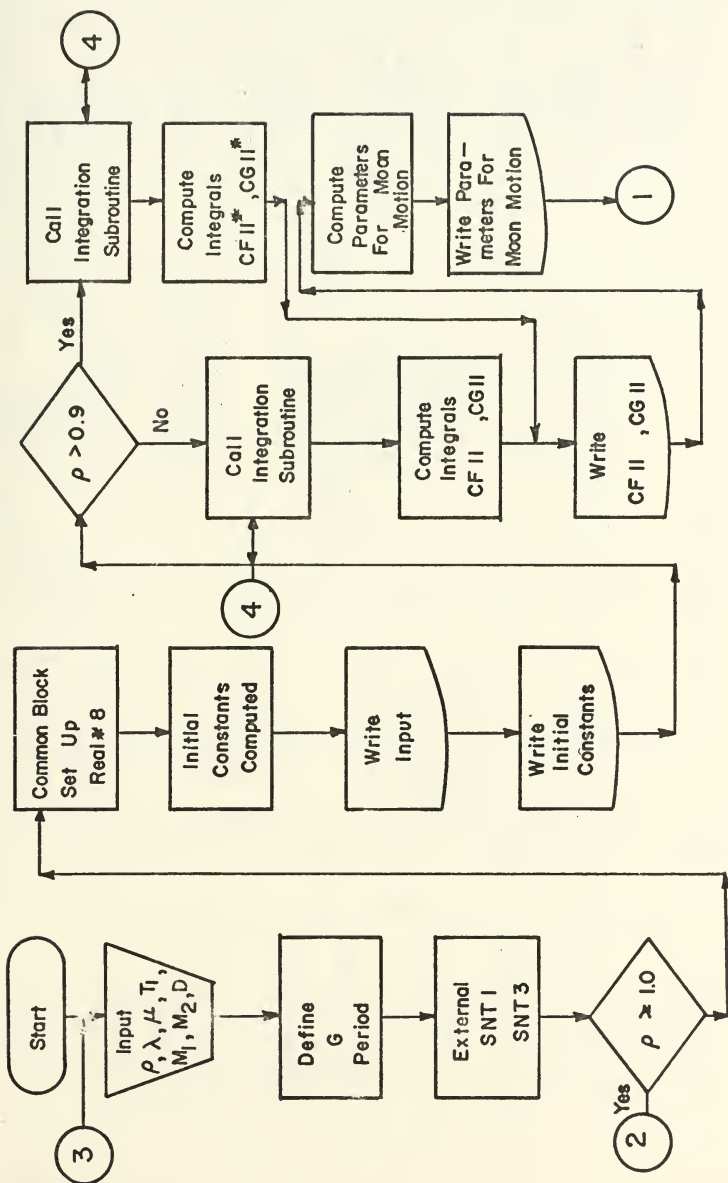


FIGURE 19. PROGRAM FLOW CHART PART A

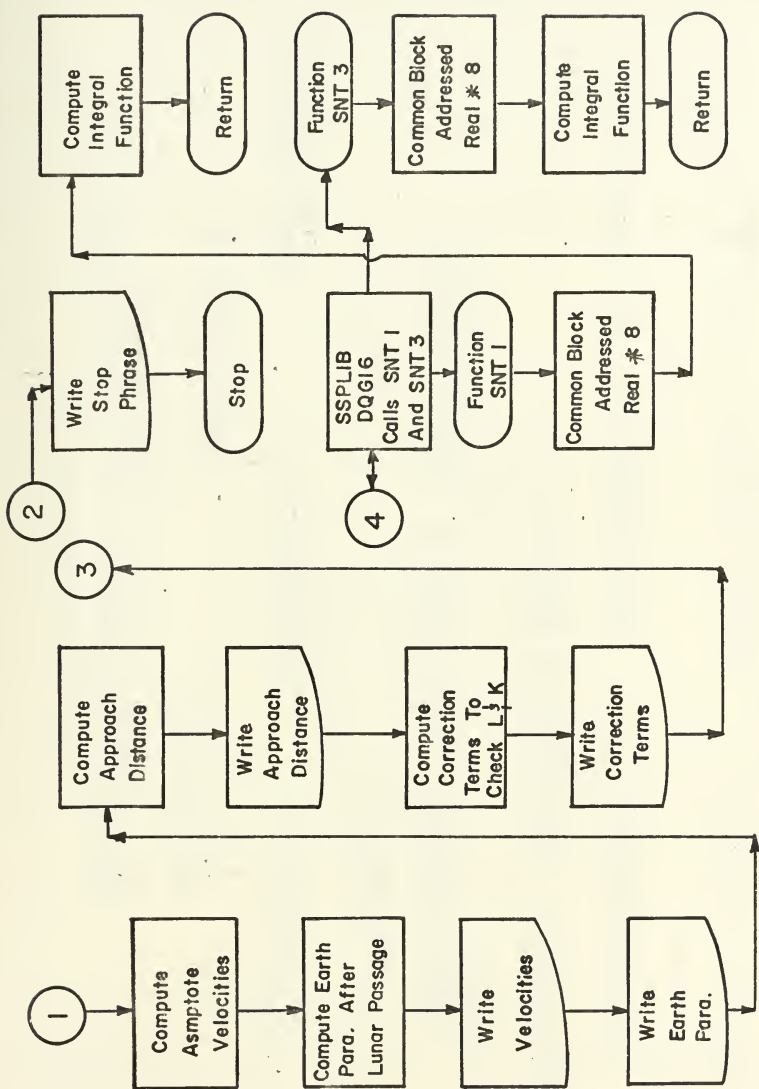


FIGURE 20 PROGRAM FLOW CHART PART B

LIST OF EQUATIONS USED IN THE IBM SOLUTION

The flowchart of Figures 17 and 18 shows how the logic of the computer solution is carried out. The following equations are those used in the solution on the computer.

INPUT

$$\rho = \left\{ \frac{(1-\mu)D}{r_p} - \frac{v_p^2 D}{2G(m_e + m_m)} \right\}^{1/2}$$

$$\lambda = \left\{ \frac{r_p v_p}{\mu D G (m_m + m_e)} \right\}^{1/2}$$

$$\mu = 0.01215 \doteq \frac{m_m}{m_e + m_m}$$

$$T_1 = \text{Phase Constant}$$

$$G = 4.967 \times 10^{-10} \text{ KM}^3/\text{KG-DAY}^2$$

$$\text{PER} = 2\pi \sqrt{\frac{D^3}{2G(m_e + m_m)}} \doteq 27.3217 \text{ DAYS}$$

$$I = \text{ORBIT INCIDENCE ANGLE}$$

$$\omega = \text{Apse Location}$$

EVALUATION OF TERMS FOR X = 1

$$T_{001} \equiv t_{00}(1) = [\sin^{-1} 2\rho(1-\rho^2)^{1/2} - 2\rho(1-\rho^2)^{1/2}] / 2^{3/2} \rho^3$$

$$T_{00P1} \equiv t_{00'}(1) = [1/2(1-\rho^2)]^{1/2}$$

$$Y_{1201} \equiv y_{12}(1) = 2^{1/2} \lambda (1-\rho^2)^{1/2}$$

$$\text{CAPT0} \equiv T_0 = t_{00}(1)$$

$$\text{CAPT12} \equiv T_{1/2} = \lambda U \cos i$$

$$\text{CAPT} \equiv T = T_0 + \rho^{1/2} T_{1/2} + \mu T_1$$

$$\text{ALPHA1} \equiv \alpha_1 = -[(1-\rho^2)^{1/2}/2 + 1/(1-\rho^2)^{1/2} - \frac{3}{2\rho} \sin^{-1} \rho] / 2^{1/2} \rho^4$$

$$U \equiv U = 1/t_{00'}(1)$$

$$Q \equiv Q = (1+U^2)^{3/2}$$

EVALUATION OF TERMS IN $\bar{x}-\bar{y}$ (MOON) SPACE

$$HBAR \equiv \bar{h} = (1+U^2)/2$$

$$XLBAR \equiv \bar{x} = [1 + \lambda^2(1+U^2 \sin^2 \bar{i})/2 + U(Y-\delta) - U(T_0 + T_1) - \xi(1,0)] / \cos \bar{i}$$

$$ABAR \equiv \bar{a} = 1/2\bar{h}$$

$$PBAR \equiv \bar{p} = U\bar{x} + (2\bar{h})^{-1/2}$$

$$QBAR \equiv \bar{q} = \bar{x} - U(2\bar{h})^{-1/2}$$

$$EPSBAR \equiv \bar{e} = \{1 + 2\bar{h}\bar{x}\}^{1/2}$$

$$TAUBAR \equiv \bar{e} = \bar{a}^{3/2} \log(\mu \bar{a}^{3/2} \bar{e} \frac{U}{2}) + [(1-\bar{a})/U]\bar{x} \cos \bar{i} + \bar{a}^{3/2} + T_1 + \delta$$

$$THEBAR \equiv \bar{\theta} = \tan^{-1}(-\bar{x}(1+U^2)^{1/2})$$

$$IBAR \equiv \bar{l} = \sin^{-1}(-\xi(1+U^2)^{1/2}/\bar{x})$$

$$OMEGA \equiv \bar{\omega} = \tan^{-1}(-1/U)$$

EVALUATION OF VELOCITY COMPONENTS

$$U1 \equiv U_1 = U$$

$$V1 \equiv V_1 = -1$$

$$W1 \equiv W_1 = 0$$

$$U11 \equiv U_{11} = (1/\bar{e}^2)(1+U^2)^{1/2} [(1+U^2)\bar{x}^2 - 1] \cos \bar{\omega} - (\frac{2}{\bar{e}^2})(1+U^2) \sin \bar{\omega} \cos \bar{x}$$

$$V11 \equiv V_{11} = (1/\bar{e}^2)(1+U^2)^{1/2} [(1+U^2)\bar{x}^2 - 1] \sin \bar{\omega} - (\frac{2}{\bar{e}^2})(1+U^2) \cos \bar{\omega} \cos \bar{x}$$

$$W11 \equiv W_{11} = (2/\bar{e}^2)(1+U^2) \sin \bar{x}$$

EVALUATION OF EARTH ORBIT PROPERTIES AFTER ENCOUNTER

$$PSI \equiv \psi = \tan^{-1}[(V_{11}+1)/U_{11}]$$

$$HEII \equiv h_{e11} = \bar{h} + V_{11} - 1/2$$

$$QEII \equiv q_{e11} = -(V_{11}+1)V_{11} + 1$$

$$PEII \equiv p_{e11} = (V_{11}+1)U_{11}$$

$$XLEII \equiv l_{e11} = V_{11} + 1$$

CLOSEST APPROACH

$$DBAR \equiv \sigma = (1/2\hbar) [(1 + 2\hbar I^2)^{1/2} - 1]$$

COMPUTE PARAMETER CHECKS OF LAGERSTROM AND KEVORKIAN

$$PARA1 = U(Y-\delta) = U(CF11 - CG11)$$

$$PARA2 = \bar{I} + UT_1 - \lambda^2/2$$

EQUATIONS USED IN THE SUBROUTINES TO OBTAIN THE MOTION INTEGRALS

$$T00Z \equiv t_{00}(z) = [\sin^{-1} z \rho (z - \rho^2 z^2)^{1/2} - z \rho (z - \rho^2 z^2)^{1/2}] / z^{3/2} \rho^3$$

$$T00PZ \equiv t'_{00}(z) = [z/2(1 - \rho^2 z)]^{1/2}$$

$$ALPHA Z \equiv \alpha(z) = -[z^{1/2}(1 - \rho^2 z)^{1/2}/2 + z^{1/2}/(1 - \rho^2 z)^{1/2} - \frac{3}{2\rho} \sin^{-1} \rho z^{1/2}] / z^{1/2} \rho^4$$

$$F0Z \equiv f_0(z) = \frac{\cos[t_{00}(z) - T_0] - z}{\{1 + z^2 - 2z \cos[t_{00}(z) - T_0]\}^{3/2}} - \cos[t_{00}(z) - T_0]$$

$$G0Z \equiv g_0(z) = \frac{\sin[t_{00}(z) - T_0]}{\{1 + z^2 - 2z \cos[t_{00}(z) - T_0]\}^{3/2}} - \sin[t_{00}(z) - T_0]$$

$$CF11 \equiv \gamma = \int_0^1 \{f_0(z) [\alpha(1) - \alpha(z)] + \frac{1}{Q(1-z)}\} dz + \frac{1}{Q}$$

$$CG11 \equiv \delta = \int_0^1 \{z t'_{00}(z) g_0(z) [\beta(1) - \beta(z)] + \frac{1}{Q(1-z)}\} dz + \frac{1}{Q}$$

THIS PROGRAM USES THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS TO SOLVE THE SINGULAR PERTURBATION PROBLEM IN EARTH-MOON SPACE. THE ANALYSIS IS BASED ON WORK BY LAGERSTROM WHICH HAS APPEARED IN THE AIAA JOURNALS.

PROGRAM TO CALCULATE MOTION OF A BODY NEAR MOON IN THE
RESTRICTED 3-BODY PROBLEM

```

INITIAL DATA
IMPLICIT REAL*8 (A-H,O-Z,$)
XMI MASS PARENT,XM2 MASS TARGET
XMR RATIO OF MASSES
XMU PERIOD OF SYSTEM
D - PLANET SEPARATION DISTANCE

```

NAMELIST/DATA1/XM1, XM2, D
NAMELIST/DATA3/RHO, XLAMB, XMU, T1
NAMELIST/DATA2/XI, SOMEGA, J

```

3  NAMELIST=(DATA1,
READ(5, DATA1)
READ(5, DATA3)
READ(5, DATA2)
IF(J.NE.O)GO TO

```

```

C
5003
C COMPUTE PARAMETERS
G=4.9672000E-10
IF(RHO,GT,1.0)GO TO 990
PER=2.00*3.1416D0*DSORT((D**3)/(G*(X1+XW2))))
COMMON/ADFILE/RHO,CAPTO,ALPHA1,Q,TOOP1,TOO1,Y1Z01,CAPT12,CAPT,U,N
EXTERNAL CSUT1

```

```

C
      COMPUTE ALL CONSTANTS TERMS FOR X=1 PART 1
      T00P1 = DSQRT(1.D0/(2.D0*(1.D0-(RHO**2))))
      T001 = (DARSIN(2.D0*RHO*DSQRT(1.D0-(RHO**2)))-(2.D0*RHO*DSQRT
      A1,Y1201=(RHO**2)/((RHO**3)*(2.D0*(3.D0/2.D0)))
      Y1201 = DSQRT(2.D0)*XLAMB*DSQRT(1.D0-(RHO**2))
      CAPT0=1001
      U = 1.D0/(COP1
      CAPT12=XLAMB**2*DCOS(XI)
      CAPT = CAPT0 + DSQRT(XMU)*CAPT12 + XMU*T1
      Q = (1.DC+U**2)**(3.D0/2.D0)

```

DRB00010
DRB00020
DRB00030
DRB00040
DRB00050
DRB00060
DRB00080
DRB00100
DRB00110
DRB00120

0RB002000
0RB002100
0RB002200
0RB002300
0RB002400
0RB002500
0RB002600
0RB002700
0RB002800
0RB002900
0RB003000
0RB003100
0RB003200
0RB003300
0RB003400
0RB003500
0RB003600
0RB003700
0RB003800
0RB003900
0RB004000
0RB004100
0RB004200
0RB004300
0RB004400
0RB004500


```

99      ALPHA1 = -( (DSQRT(1,D0-(RHO**2))/2,D0) + (1,D0/(DSQRT(1,D0-
A(RHO**2)))) - (3,D0/(2,D0*RHO))*DARSIN(RHO))/(RHO**4)*
WRITE(6,99)
FORMAT(11,'*****')
A('*****')
WRITE(6,100)
WRITE(6,12004)X1,X2,D,PER,G
FORMAT(1,D0,T20,MASS,PARENT =E12.6,KMS,T50,PERIOD=E12.6,DAYS/
A,KGS,/,T20,SEP DIS=E12.6,KMS,T50,PERIOD=E12.6,DAYS/
B,T20,GRAVITY CONSTANT=E12.6,KMS**2/DAY**2*KG/)
WRITE(6,101)RHO,XMU,XLAMB,T1,X1,SOMEGA
WRITE(6,103)TOOP1,TOOP1,T001,Y1201,CAPTO,CAPTI2,CAPT,U,Q,ALPHA1
WRITE(6,1000)
IF(RHO,G1,Q,90000) GO TO 4000
CALL DQG16(C,D0,1,D0,SN1,Y1)
CFL1=Y1+1,D0/Q
CALL DQG16(C,D0,1,D0,SN3,Y3)
CG11=Y3+1,D0/Q
GO TO 4001

C4000  CALL DQG16(C,D0,1,D0,SN5,Y5)
GSS=(1,D0/RHO**4)*Y5
CALL DQG16(C,D0,1,D0,SN6,Y6)
GS=Y6+(1,D0/(3,D0-2,D0*(RHO**2)))*(3,D0/2,D0))
CALL DQG16(C,D0,1,D0,SN7,Y7)
DS=Y7+(1,D0/(3,D0-2,D0*(RHO**2)))*(3,D0/2,D0))
CFL1=(1,D0/U)*GSS+GS-(1,D0/(RHO**4))*(3,D0-2,D0*(RHO**2))
A*(3,D0/2,D0))*DLOG(U*(DSQRT((U**2)+2,D0)-U))
CG11=DS+(1,D0/(3,D0-2,D0*(RHO**2)))*(3,D0/2,D0))*DLOG(
AU*(DSQRT((U**2)+2,D0)-U))

C4001  WRITE(6,104) CFL1,CG11
C4002
C4003  COMPUTE FIRST ORDER MOTION ONLY
HBAR=(1,D0+U**2)/2,D0
ABAR=1,D0/(1,D0+U**2)
SPSI10=Q,D0
ZETA10=XI*DSIN(SOMEGA)
OMEGA=DATAN((-ZETA10*DSQRT(1,D0+U**2))/(1,D0+(XLAMB**2))*
XIBAR=DATAN((-ZETA10*DSQRT(1,D0+U**2))/(1,D0+(XLAMB**2))*
1(1,D0+U**2))*DSIN(XI)**2)/2,D0 +U*(CFL1-CG11)-U*(CAPTO+
2(1)-SPSI10))
XLBAR=(1,D0+(XLAMB**2))*(1,D0+(U**2))*DSIN(XI)**2)/2,D0
1+U*(CFL1-CG11)-U*(CAPTO+T1)-SPSI10)/DCOS(XIBAR)
PBAR=U+XLBAR+1,D0/DSQRT(2,D0*HBAR)

ORR00450
ORR00470
ORR00480
ORR00490
ORR00500
ORR00510
ORR00520
ORR00530
ORR00540
ORR00550
ORR00560
ORR00590
ORR00600
ORR00610
ORR00620
ORR00640
ORR00650
ORR00670
ORR00690
ORR00740
ORR00760
ORR00770
ORR00790
ORR00800
ORR00810
ORR00820
ORR00830
ORR00840
ORR00850
ORR00860
ORR00900
ORR00910
ORR00930
ORR00940

```



```

QBAR=XLBAR-U/DSQRT(2.0D*HBAR)
EPSBAR=DSQRT(1.0D+2.0D*HBAR*(XLBAR**2))
THEBAR=DATAN(-XLBAR*DSQRT(1.0D+U**2))
TAUBAR=(A7AR**3.0D/2.0D)*DLOG(XMU*(ABAR**3.0D/2.0D))
1*EPSBAR*(U/2.0D)+(1.0D-ABAR)/U*XLBAR*DCOS(XIBAR)
2*ABAR*(3.0D/2.0D)*1+CG1
THEBAR=THEBAR*(180.0D/3.1415927D)
WRITE(6,850)
WRITE(6,851)HBAR,XLBAR,ABAR,PBAR,QBAR,EPSBAR,THEBAR,TAUBAR,OMEGB
1,XIBAR
C COMPUTE INCMING VELOCITIES UI AND VI
UI=-1.0D
VI=0.0D
C COMPUTE OUTGOING COMPONENTS OF VELOCITY
UIP=(1.0D+(EPSBAR**2))*DSQRT(1.0D+U**2)*
1((1.0D+U**2)*(XLBAR**2)-1.0D)
VIIP=(2.0D/(EPSBAR**2))*(1.0D+U**2)
UII=UIP*DCOS(OMEGB)-VIIP*DSIN(OMEGB)*DCOS(XIBAR)
VII=VIIP*DSIN(OMEGB)+VIIP*DCOS(OMEGB)*DCOS(XIBAR)
WII=VII*PSIN(XIBAR)
THE INITIAL DIRECTION AFTER MOON RELATIVE TO EARTH
PSI=DATAN(VII+1.0D/UII)
PSID=PSI*180.0D/3.1415927D
C KEPLERIAN INTEGRALS FOR MOTION RELATIVE TO EARTH AFTER MOON PASSAGE
HEII=HBAR*VII-0.5D
XLEII=VII+1.0D
PEII=-VII*(VII+1.0D)
QLEII=-(VII+1.0D)*VII+1.0D
C PRINT TARGET BODY DATA
WRITE(6,3000)UI,VI,WI,UII,VII,WII
3000 FORMAT(1H0,1X,*,**,**,INCOMING AND OUTGOING VELOCITIES ***/
1,*,UI=,E12.6,T23,VI=,E12.6,T40,WI=,E12.6,T15,
2,*,VII=,E12.6,T23,VII=,E12.6,T40,WII=,E12.6,T15)
C
WRITE(6,3001)PSID,HEII,XLEII,PEII,QLEII
3001 FORMAT(1H0,1X,*,PSIDEGREE=,E12.6,T40,HEII=,E12.6,T60,XLEII=,E12.6,
1,*,PEII=,E12.6,T40,QLEII=,E12.6,T15)
DBAR=D9AR*XMU*D
DBAR=D9AR*XMU*D
DEII=(1.0D/2.0D*HEII)*(1.0D-DSQRT(1.0D+2.0D*HEII*(XLEII**2)))
DEIKM=DEII*D
WRITE(6,3002)DBAR
3002 FORMAT(1H0,1X,*,DBAR=,E12.6,T15)
PARA1=U*(PEII-CG1)
PARA2=XLBAR*U*VI-(XLEBAR**2)/2.0D
WRITE(6,3008)PARA1,PARA2
3008 FORMAT(1H0,1X,*,U(GAMMA-DELTA)=,E15.8,5X,LBAR*U*VI-(XLEBAR**2)/2.0D,
1,*,PARA1=,E15.8,5X,PARA2=,E15.8,5X)
C ***TARGET ORBIT PARAMETERS***
FORMAT(1X,*,
3002
3003
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3005
3006
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3008
3009
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FORMULA, WHICH INTEGRATES POLYNOMIALS UP TO DEGREE 31
EXACTLY, FOR REFERENCE, SEE
V.I. KRYLOV, APPROXIMATE CALCULATION OF INTEGRALS,
MACMILLAN, NEW YORK/LONDON, 1962, PP.100-111 AND 337-340.

ORR01920
ORR01930
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SUBROUTINE DQGL6(XL,XU,FCT,Y)

DOUBLE PRECISION XL,XU,Y,A,B,C,FCT

```

A=5D0*(XU+XL)
B=XU-XL
C=.9470046749582497D0*B
Y=.13576229705877047D-1*(FCT(A+C)+FCT(A-C))
C=.47228751133681629D0*B
Y=Y+.31126761969323946D-1*(FCT(A+C)+FCT(A-C))
C=.43281560119391587D0*B
Y=Y+.47579255841246392D-1*(FCT(A+C)+FCT(A-C))
C=.37770220417750152D0*B
Y=Y+.6231448567766936D-1*(FCT(A+C)+FCT(A-C))
C=.30893812220132187D0*B
Y=Y+.7479796240828837D-1*(FCT(A+C)+FCT(A-C))
C=.2290083882891369D0*B
Y=Y+.8457825969750127D-1*(FCT(A+C)+FCT(A-C))
C=.14080177538962946D0*B
Y=Y+.813017C52246179D-1*(FCT(A+C)+FCT(A-C))
C=.4750625491881872D0-1*B
Y=B*(Y+.9472530522753425D-1*(FCT(A+C)+FCT(A-C)))
RETURN
END

```

ORR01990
ORR02000
ORR02010
ORR02020
ORR02030
ORR02040
ORR02050
ORR02060
ORR02070
ORR02080
ORR02090
ORR02100
ORR02110
ORR02120
ORR02130
ORR02140
ORR02150
ORR02160
ORR02170
ORR02180
ORR02190
ORR02200
ORR02210
ORR02220
ORR02230

```

FUNCTION SNT1(Z)
IMPLICIT REAL*8(A-H,O-Z,$)
COMMON/ADOLPHO,CAPTO,ALPHA1,Q,T00P1,T001,Y1201,CAPT12,CAPT,U,N
N=N+1
T00Z = (DARSIN(2.D0*RHO*DSQRT(Z-(RHO**2)*(Z**2)))-2.D0 * RHO*
ADSORT(Z-(RHO**2)*(Z**2)))/((RHO**3)*(2.D0**3*(3.D0/2.D0)))
ALPHAZ = -((DSQRT(Z))*DSQRT(1.D0-(RHO**2)*(Z)/2.D0) + (DSQRT(Z)/
ADSORT(1.D0-(RHO**2)*Z)) - ((3.D0/2.D0*RHO))*DARSIN(RHO*DSQRT
B(Z)))/(RHO**4)*DSQRT(2.D0)
FOZ = ((DCOS(T00Z-CAPTO)-Z)/((1.D0+(Z**2)-2.D0*Z*DCOS(T00Z-
CAPTO)))*(3.D0/2.D0)) - DCOS(T00Z-CAPTO)
SNT1=FOZ*(ALPHA1-ALPHAZ)+(1.D0/(O*(1.D0-Z)))

```

ORR02240
ORR02250
ORR02260
ORR02270
ORR02280
ORR02290
ORR02300
ORR02310
ORR02320
ORR02330
ORR02340
ORR02350


```

RETURN
END
OR802360
OR802370

FUNCTION SNT3(Z)
IMPLICIT REAL*8(A-H,O-Z,$)
COMMON/ADOLF/RHO,CAPTO,ALPHA1,Q,TOOP1,TOOL,Y1201, CAPT12,CAPT,U,N
N=N+1
BETA1 = -(1.00/TOOP1)
TCOPZ = DSQRT(Z*(2.00*DO*(1.00-(RHO**2)*Z)))
BETAZ = -(1.00/TOOPZ)
TOOZ = -(DARSIN(2.00*RHO*DSQRT(Z-(RHO**2)*(Z**2))) - 2.00 * RHO*
ADSQRT(Z-(RHO**2)*(Z**2)))/((RHO**3)*(2.00**3.00/2.00))
CGZ = (DSIN(TOOZ-CAPTO)/(1.00+(Z**2)-2.00*Z*DCOS(TOOZ-CAPTO))**
A(3.00/2.00)) - DSIN(TOOZ-CAPTO)
SNT3 = Z*TCOPZ*CGZ*(BETA1-BETAZ) + (1.00/(Q*(1.00-Z)))
RETURN
END
OR802490
OR802490
OR802500
OR802500
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OR802570
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OR802590
OR802600
OR802610

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13. ABSTRACT A program was written for the IBM 360 system of the Naval Postgraduate School Computer Facility to apply the methods of singular perturbation theory to earth-moon trajectories. Verification of results against previous work was carried out and it was found that agreement could be attained only for energy parameter values of 0.707 or less. No solution for higher values could be found. The analysis of three dimensional orbits was then conducted within this restricted range to show the merit of singular perturbation theory as an initial design tool.			

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
PERTURBATIONS						
EARTH-MOON SYSTEM						
MATCHED ASYMPTOTIC EXPANSIONS						
TRAJECTORIES						

Thesis

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Singular perturbation
problems in earth-moon
space.

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